Choose to work one and only one of the following problems.

1. Prove the Chebyshev's theorem on polynomial approximations: Let $f(x)$ be a continuous function over the interval $[a, b]$. Let $\mathcal{P}_n$ denote the set of polynomials of degree $\leq n$. Then $p^*(x) \in \mathcal{P}_n$ is the best approximation to $f(x)$ in the sense that

$$\|p^* - f\|_\infty = min_{p \in \mathcal{P}_n} \|p - f\|_\infty$$

if and only if there exist $N \geq n + 2$ points in $[a, b]$,

$$x_1^* < x_2^* < \ldots < x_N^*,$$

such that

$$p^*(x_k^*) - f(x_k^*) = (-1)^k \|p^* - f\|_\infty, \quad k = 1, 2, \ldots, N.$$

2. Implement the Remez's algorithm based on the idea outlined in class into a computer program. Recall that you need to do two subroutines inside:

   (a) You need to solve a linear system to update the coefficients of the approximate polynomial.

   (b) You need to locate new alternant points by solving some optimization problems.

   Search around the Matlab to use any available packages (with modifications) for the above two purposes. Try out your code on the function

$$f(x) = \cos \frac{1}{4}\pi x, \quad x \in [-1, 1]$$

   with a polynomial of degree ≤ 7. Compare your answer against the one known in the literature (but, alas, I have purposefully changed some decimals waiting for your correction):

$$p^*(x) \approx .9999999824 - .3084242736x^2 + .0158499453x^4 - .0003188205x^6$$

   Report your coefficients up to the 15th digits.