1. It is known that the integral

$$\int_0^1 \frac{2}{2+\sin 10\pi x}dx = \frac{2}{\sqrt{3}} \approx 1.154700538379251529018297561003914.....$$

Describe one numerical procedure to evaluate the integral with accuracy up to the tenth digit. The description should be specific on all parameters involved so as to guarantee (Suppose that you do not have any a priori knowledge about the final answer, how can be sure about the accuracy of your numerical answer?) accuracy desired.

2. Denote the two-parameter Hankel transform of the function

$$f(x) = \frac{x}{(x^2+1)^{11/6}} \tag{1}$$

by

$$H(a,b) = \int_0^\infty f(x)J_0(ax)J_0(bx)\,dx \tag{2}$$

where a, b and x are variable nonnegative real numbers, and $J_0(x)$ is the zero-order Bessel function of the first kind, i.e.,

$$J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2}\left(\frac{x}{2}\right)^{2j}. \tag{3}$$

In an application to the adaptive optics, it is desired to effectively construct an $n \times n$ matrix

$$H(\vec{a},\vec{b}) = [H(a_i,b_j] \tag{4}$$

for any n values of $\vec{a}$ and $\vec{b}$. Design an algorithm using quadratures to evaluate (2) efficiently and accurately. Again, explain why you are sure that your algorithm is producing an accurate answer up to, say, the sixth digit. Also, gauze the CPU time needed for computation. Report your answers for the 6×6 matrix $H(0:0.2:1, 0:0.2:1)$ where each entry is accurate up to the fifth digit. (*Note: There are many other ways to describe the Bessel function. For example, the zero-order Bessel function can also be expressed in the integral form:*

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\,d\theta. \tag{5}$$

You might want to do some readings before you jump in.)