1. (The distance to the nearest singular matrix) Let P denote the set of singular matrices in $R^{n\times n}$. Let A be a fixed nonsingular matrix. Let $||\cdot||$ denote an arbitrary vector norm and $||A||$ the induced matrix norm. Follow the procedures below to show that the minimum distance from the matrix A to the set P is given by

$$dist(A,P) = ||A^{-1}||^{-1}. \tag{1}$$

   (a) Let B be a singular matrix. Show that

$$||A-B|| \geq ||A^{-1}||^{-1}. \tag{2}$$

   (b) Introducing the dual norm:

$$||y^T||_D := \sup_{x\neq 0} \frac{y^T x}{||x||}, \tag{3}$$

   show that there exist vectors x and y such that

$$||x|| = ||y^T||_D = 1 \tag{4}$$

   and

$$y^T A^{-1} x = ||A^{-1}||. \tag{5}$$

   (c) Define

$$\delta A := -||A^{-1}||^{-1} x y^T. \tag{6}$$

   Show that $A+\delta A$ is singular and that $||\delta A|| = ||A^{-1}||^{-1}$.

2. (Backward error analysis for triangular systems)

   (a) Show that if a lower triangular system

$$Lx = b \tag{7}$$

   is solved in floating-arithmetic, then there exists a lower triangular matrix δL such that the computed solution $\overline{x}$ satisfies the system

$$(L+\delta L)\overline{x} = b. \tag{8}$$

   (b) Give an estimate of $||\delta L||$.

3. (Matrix inversion by rank annihilation)

   (a) Let u and v be column vectors and A a nonsingular matrix. Verify that

$$(A+uv^T)^{-1} = A^{-1} - \frac{(A^{-1}u)(v^T A^{-1})}{1+v^T A^{-1} u}. \tag{9}$$

(b) Let

$$B = D + \sum_{i=1}^{m} u_i v_i^T \tag{10}$$

where D is a nonsingular diagonal matrix. Define

$$C_k := (\sum_{i=1}^{k} u_I v_i^T + D)^{-1}. \tag{11}$$

Use the results from part(a) to prove that

$$C_{k+1} = C_k - \frac{(C_k u_{k+1})(v_{k+1}^T C_k)}{1 + v_{k+1}^T Ck u_{k+1}}. \tag{12}$$

(c) Use the result of part(b) to deduce an algorithm for calculating B^{-1} with B as given in (10).