Important: There are three categories of problems. Answer one and only one problem from each category.

Category A

A1. (a) Show that if a lower triangular system

$$Lx = b \tag{1}$$

is solved in floating-arithmetic, then there exists a lower triangular matrix δL such that the computed solution $\overline{x}$ satisfies the system

$$(L + \delta L)\overline{x} = b. \tag{2}$$

(b) Give an estimate of $||\delta L||$.

A2. Let A be a real symmetric positive definite matrix. Prove the following statements:

(a) There exists a unit lower triangular matrix L and an upper triangular matrix U such that $A = LU$. (Note: You must NOT assume the existence of any other type of decomposition before proving it first.)

(b) $\max\limits_{1\le i,j\le n} |u_{ij}| \le \max\limits_{1\le i,j\le n} |a_{ij}|$

(c) $u_{ii} > 0$ for $i = 1, \ldots, n$.

Category B

B1. Let $A \in R^{n\times n}$.

(a) Describe the power method, the inverse power method and the Rayleigh quotient iteration method. Explain what each of these methods does in solving the eigenvalue problem of A.

(b) Describe the QR decomposition of the matrix A. Explain the relationship between the Gram-Schmidt orthogonalization process and the QR decomposition of the matrix A.

(c) Describe the basic QR algorithm and the explicit single-shift QR algorithm.

(d) What is the Wilkinson shift? Give heuristic reasons why this shift helps to find eigenvalues.

(e) Let p be a chosen integer satisfying $1 \le p \le n$. Given an $n \times p$ matrix Q_0 with orthonormal columns, consider the iteration

$$\begin{aligned} Z_k &= AQ_{k-1} \\ Q_k R_k &= Z_k \text{ (QR factorization)} \end{aligned}$$

for $k = 1, 2, \ldots$. Explain why this iteration can usually be used to compute the p largest eigenvalues of A in absolute value. How then should you modify the iteration when the p smallest eigenvalues of A in absolute value are needed.

B2. Let A be a real symmetric positive definite matrix and assume it is known that

$$0 < \alpha \le \lambda \le \beta$$

for λ any eigenvalue of A. For iterative methods in solving $Ax = b$, let $x^{(j)}$ and $r^{(j)} = b - Ax^{(j)}$ denote an iterate and its corresponding residual, respectively.

(a) Richardson's method is given by

$$x^{(j+1)} = x^{(j)} + r^{(j)}. \tag{3}$$

Does this method always converge ? State why or why not.

(b) Consider a modified form of Richardson's method

$$x^{(j+1)} = x^{(j)} + \gamma r^{(j)}. \tag{4}$$

Determine values of γ for which method (4) always converges to the solution of the system $Ax = b$. Also, determine γ^*, the optimal value of γ for which method (4) has the highest rate of convergence. With this value of γ^*, what can be said about the number of iterations in the special case of $\alpha = \beta$?

(c) Now consider a modified form of Richardson's method

$$x^{(j+1)} = x^{(j)} + \gamma^{(j)} r^{(j)}. \tag{5}$$

in which the parameter $\gamma^{(j)}$ is allowed to vary with each iteration. Determine the value of $\gamma^{(j)}$ for which the Euclidean norm of $r^{(j+1)}$ is minimized. (Note: $\gamma^{(j)}$ should only depend on A, b, $x^{(j)}$ and $r^{(j)}$.) Choosing $\gamma^{(j)}$ in this manner, will the method converge? What can be said about the number of iterations in the special case of $\alpha = \beta$?

(d) Which of the two methods given by (4) and (5) is preferable and why ?

Category C

C1. Let P denote the set of singular matrices in $R^{n\times n}$. Let A be a fixed nonsingular matrix. Let $||\cdot||$ denote an arbitrary vector norm and $||A||$ the induced matrix norm. Follow the procedures below to show that the minimum distance from the matrix A to the set P is given by

$$dist(A, P) = ||A^{-1}||^{-1}. \tag{6}$$

(a) Let B be a singular matrix. Show that

$$||A - B|| \geq ||A^{-1}||^{-1}. \tag{7}$$

(b) Introducing the dual norm:

$$||y^T||_D := \sup_{x\neq 0} \frac{y^T x}{||x||},$$

show that there exist vectors x and y such that

$$||x|| = ||y^T||_D = 1 \tag{8}$$

and

$$y^T A^{-1} x = ||A^{-1}||. \tag{9}$$

(c) Define

$$\delta A := -||A^{-1}||^{-1} x y^T. \tag{10}$$

Show that $A + \delta A$ is singular and that $||\delta A|| = ||A^{-1}||^{-1}$.

C2. Let A be an invertible real $n \times n$ matrix. Given $b \in R^n$, let x be the exact solution of the linear system $Ax = b$, and assume there is also an approximate solution $\bar{x}$. Let $r = b - A\bar{x}$.

(a) Determine a bound on the relative error of the approximate solution in the max–norm ($\|x - \bar{x}\|_\infty / \|x\|_\infty$) in terms of $\|b\|_\infty$, $\|r\|_\infty$, and $K_\infty(A)$, the condition number of A with respect to the $\|\cdot\|_\infty$ norm for matrices.

(b) Let A be given by

$$A = \begin{pmatrix} 10^8 & 10^{-8} \\ 10^{-8} & 10^{-8} \end{pmatrix}.$$

Estimate $K_\infty(A)$. What does this tell you about approximate solutions to $Ax = b$ when the residual r is small ?

(c) Let D be the diagonal matrix given by

$$D = \begin{pmatrix} 10^{-4} & 0 \\ 0 & 10^4 \end{pmatrix}$$

and define $\tilde{A} = DAD$. Estimate $K_\infty(\tilde{A})$.

(d) Describe how the original equation $Ax = b$ may be transformed into the new equation $\tilde{A}y = \tilde{b}$. Compare and contrast these two equations.

C3. (a) Describe the Jacobi, Gauss-Seidel, and SOR iterative methods for solving $Ax = b$.

(b) Prove that the Jacobi method is convergent if A is strictly diagonally dominant.

(c) Show that both Jacobi and Gauss-Seidel are convergent if A is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Also, compare the number of iterations required by these methods.