1.) Let A be an invertible real $n \times n$ matrix. Given $b \in R^n$, let x be the exact solution of the linear system $Ax = b$. Assume this system is also solved by Gaussian elimination (LU-decomposition) with finite precision arithmetic which gives an approximate solution $\bar{x}$. Let $r = A\bar{x} - b$. Let C be an approximation to the inverse of A and assume that $\delta = \|I - CA\| < 1$. Here, $\|\cdot\|$ denotes a vector norm on R^n and the associated induced matrix norm.

(a.) Prove that

$$\frac{\|Cr\|}{1+\delta} \le \|x - \bar{x}\| \le \frac{\|Cr\|}{1-\delta}.$$

(b.) Show that if the factors of the LU-decomposition of A are available, then computing Cr requires $O(n^2)$ multiplications.

2.) Consider the iterative method

$$u^{(n+1)} = Gu^{(n)} + y \tag{1}$$

and the related iterative method

$$u^{(n+1)} = \gamma[Gu^{(n)} + y] + (1-\gamma)u^{(n)} \tag{2}$$

for the solution of a linear system $Ax = b$. Assume that the iteration matrix G is symmetric and that its eigenvalues satisfy

$$-c = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 0 \tag{3}$$

for some positive constant c.

(a.) For what values of c will the method (1) converge to the solution of the linear system?

(b.) Determine values of γ for which method (2) converges to the solution of the system for all values of $c > 0$.

(c.) Determine γ^*, the optimal value of γ for which method (2) has the highest rate of convergence.

(d.) Using the value of γ^* found above in method (2) compare the rates of convergence of the two methods assuming that the constant c is such that both methods will converge.

3.) Let A be the real $n \times n$ matrix

$$A = \begin{pmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{pmatrix}.$$

(a.) Find an upper bound on $K_2(A)$, the condition number of A with respect to the matrix norm induced by the Euclidean vector norm.

(b.) Let J denote the iteration matrix in the Jacobi iterative method for solving the linear system $Ax = b$. Find an upper bound for the spectral radius of J which shows $\rho(J) < 1$, and hence the Jacobi method will converge.

(c.) Determine the least number of Jacobi iterations so that the error of any initial geuss will be reduced by a factor of 10^{-5}. Compute the total amount of multiplications required by this number of Jacobi iterations for the solution of $Ax = b$.

(d.) With respect to work and error, compare the Jacobi iterative method to Gaussian elimination for the solution of the tridiagonal system $Ax = b$.