Answer at least one of the first two questions and a total of 4 problems.

1. Let A be a real $n \times n$ matrix.
   (a) Show that A is symmetric and positive definite if and only if there exists a nonsingular lower triangular matrix L such that
   $$A = LL^T$$
   (b) Show that if the diagonal elements of L are taken to be positive, then the matrix L is unique.
   (c) Assume that the components of $A = (a_{ij})$ satisfy $|a_{ij}| \le 8$ for $i, j = 1, \ldots, n$. Show that the components of $L = (l_{ij})$ satisfy $|l_{ij}| \le 4$.

2. (a) Define what is meant by floating-point arithmetic.
   (b) Show that if a lower triangular system
   $$Lx = b$$
   is solved in floating-point arithmetic, then there exists a lower triangular matrix δL such that the computed solution $\overline{x}$ sat isfies the system
   $$(L + \delta L)\overline{x} = b.$$
   (c) Give an estimate of $\|\delta L\|$. (You may choose any norm you want.)

3. Let A be the real $n \times n$ matrix
$$A = \begin{pmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{pmatrix}.$$
   (a) State and prove any form of Gerschgorin Circle Theorem which defines a set that contains all the eigenvalues of a matrix.
   (b) Using the Gerschgorin Circle Theorem, determine bounds on the spectrum of A and an upper bound on $K_2(A)$, the condition number of A with respect to the matrix norm induced by the Euclidean vector norm.
   (c) Is this a well–conditioned matrix ? Give specific reasons to support your answer.
   (d) Which of the following methods will converge to the solution of the linear system $Ax = b$ for any arbitrary initial guess: Jacobi, Gauss–Seidel, SOR, Conjugate Gradient method. Give reasons to support your answers.

4. Solve the eigenvalue problem for tridiagonal matrix
$$A = \begin{pmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{pmatrix}, \quad bc > 0$$
   according to the following steps:

(a) Assuming $v_0 = v_{n+1} = 0$, show that the eigenvalue problem $Av = \lambda v$ with $v = [v_1, \dots, v_n]^T$ is equivalent to solving the finite difference equations

$$cv_{j-1} + (a - \lambda)v_j + bv_{j+1} = 0$$

for $j = 1, \dots, n$.

(b) Assuming $v_j = m^j$, show that m solves the quadratic equation

$$bm^2 + (a - \lambda)m + c = 0. \tag{1}$$

(c) Let m_1 and m_2 denote the distinct roots of (1). Then by principle of superposition, it follows that $v_j = \alpha m_1^j + \beta m_2^j$ for some constants α and β. Using the fact that $v_0 = v_{n+1} = 0$, show that $m_1 = (\frac{c}{b})^{1/2} \exp(\frac{s\pi i}{n+1})$ and $m_2 = (\frac{c}{b})^{1/2} \exp(\frac{-s\pi i}{n+1})$ for $s = 1, \dots, n$.

(d) Show that the eigenvalues of A are given by

$$\lambda_s = a + 2\sqrt{bc}\cos(\frac{s\pi}{n+1}).$$

5. In the following, P is a real $n \times n$ matrix while w and y are vectors in R^n.

(a) Suppose P is of the form

$$P = I - \alpha w w^T \tag{2}$$

where w is a unit vector ($w^T w = 1$). Determine the value of α so that P is an orthogonal matrix.

(b) Given a vector y, determine the vector w and the value of k in order to have

$$Py = (k, 0, 0, \dots, 0)^T$$

where P is an orthogonal matrix of the form (2).

(c) Describe an algorithm that uses the above transformation to solve a linear system of equations $Ax = b$.

6. Consider the iterative method

$$u^{(n+1)} = Gu^{(n)} + y \tag{3}$$

and the related iterative method

$$u^{(n+1)} = \gamma[Gu^{(n)} + y] + (1 - \gamma)u^{(n)} \tag{4}$$

for the solution of a linear system $Ax = b$. Assume that the iteration matrix G is symmetric and that its eigenvalues satisfy

$$-c = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 0$$

for some positive constant c.

(a) For what values of c will the method (3) converge to the solution of the linear system?

(b) Determine values of γ for which method (4) converges to the solution of the system for all values of $c > 0$.

(c) Determine γ^*, the optimal value of γ for which method (4) has the highest rate of convergence.

(d) Using the value of γ^* found above in method (4) compare the rates of convergence of the two methods assuming that the constant c is such that both methods will converge.

7. Consider the least squares approximation of a given real $n \times n$ matrix A by a multiple of a symmetric matrix of rank one. That is,

$$\begin{array}{ll} \text{Minimize} & f(x,c) = \|A - cxx^T\|^2 \\ \text{subject to} & x \in R^n,\ x^T x = 1 \text{ and } c \in R \end{array} \qquad (5)$$

where $\|\cdot\|$ means the Frobenius norm

$$\|A\| = \left(\sum_{i,j} a_{ij}^2 \right)^{1/2}.$$

(a) Using the fact that the inner product $< A, B >= \text{trace}(AB^T)$ generates the Frobenius norm, show that for a given normalized x, the objective function $f(x,\cdot)$ is minimized by $c =< A, xx^T >$.

(b) Show that the least squares problem (5) is equivalent to

$$\begin{array}{ll} \text{Maximize} & g(x) = | < A, xx^T > | \\ \text{subject to} & x \in R^n \text{ and } x^T x = 1. \end{array}$$

(c) Show that if A is symmetric and positive definite, then the optimal solution is that c^* = the largest eigenvalue of A, and x^* = the normalized eigenvector corresponding to c^*.

8. Consider the two-point boundary value problem

$$\begin{aligned} -(K(x,u)u_x)_x &= f(x), \\ u(0) &= 0 = u(1). \end{aligned} \qquad (6)$$

(a) State a finite difference method for solving (6).

(b) State Newton's method for the problem in (a). Be sure to explicitly state the Jacobian.

(c) Let $F : R^n \longrightarrow R^n$ and $F(x^*) = 0$. Assume that F is continuously differentiable in a neighborhood of x^* and that $F'(x^*)$ is nonsingular. Prove that Newton's method converges if the starting point is sufficiently close to x^*.