Important: There are three categories of problems. Answer one and only one problem from each category.

Category A

A1. Describe the Gauss-Seidel iterative method for solving $Ax = b$. Prove that the method is convergent if A is a real symmetric positive definite matrix.

A2. In the following, A is a real symmetric positive definite $n \times n$ matrix, and lower case letters denote real vectors of length n.

(a) Show that the problem of solving $Ax = b$ is equivalent to minimizing the functional $F(u)$ where

$$F(u) = \frac{1}{2}u^T Au - u^T b.$$

(b) Let p and v be given fixed vectors. (Think of v as an approximation of the solution x, and p is a direction you wish to follow in order to improve the approximation.) Determine the value of α so that $r^T r$ is as small as possible where

$$r = Aw - b$$

and

$$w = v + \alpha p.$$

(c) Suppose instead of minimizing $r^T r$, we wish to make $F(w)$ as small as possible where w has the same form as above. Determine the value of α that accomplishes this goal.

(d) Compare the 2 strategies in parts (b) and (c). In particular, state which one is related to the conjugate gradient algorithm.

A3. Consider the two-point boundary value problem

$$\begin{aligned} -u_{xx} &= f(x, u) \\ u(0) &= 0 \\ u(1) &= 0. \end{aligned}$$

(a) Describe the resulting nonlinear algebraic system when a finite difference method, say, the central difference quotient, is used.

(b) Apply Newton's algorithm to this problem. State $F(u)$, $F'(u)$ and the algorithm.

(c) Outline the proof of convergence, define local convergence, and define quadratic convergence.

Category B

B1. Let A be the real symmetric matrix given by

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}.$$

(a) Define the condition number $K_2(A)$ of a matrix corresponding to the spectral norm and find the condition number for this particular matrix.

(b) Define what it means for a matrix to be real symmetric positve definite. Is the given matrix A real symmetric positive definite ?

(c) For real symmetric positive definite matrices, compare the use of Gaussian elimination with and without pivoting. In particular explain why or why not pivoting is necessary or useful.

(d) Let A be nonsingular and consider the system of linear equations $Ax = b$. Suppose we compute an approximate solution y to this problem which is the exact solution of the perturbed problem $(A + \Delta A)y = b + \Delta b$ where it is known that

$$\begin{aligned} \|\Delta A\| &\leq 0.1 \quad \|A\|, \\ \|\Delta b\| &\leq 0.1 \quad \|b\|, \\ \|\Delta A\| \; \|A^{-1}\| &\leq 0.2, \end{aligned}$$

determine a bound on the relative error of the approximation

$$\frac{\|y - x\|}{\|x\|}$$

where x is the exact solution.

B2. Consider the least squares (LS) problem

$$\min_{x \in R^n} \|Ax - b\|$$

where $A \in R^{m \times n}$, rank(A) $= n \leq m$.

(a) Show that there is a unique least squares solution x_{LS} and that x_{LS} solves the normal equation

$$A^T A x_{LS} = A^T b.$$

(b) Describe the QR decomposition and the SVD decomposition of the matrix A.

(c) Explain how the least squares solution x_{LS} can be obtained from either the QR decomposition or the SVD decomposition of A.

(d) Sketch a numerical procedure to compute either the QR decomposition or the SVD decomposition of A.

(e) Suppose now $\text{rank}(A) = r < n$. Explain how the unique solution with minimum 2-norm can be obtained from the SVD.

Category C

C1. Let $A \in R^{n \times n}$.

(a) Describe the power method, the inverse power method and the Rayleigh quotient iteration method. Explain what each of these methods does in solving the eigenvalue problem for A.

(b) Describe the basic QR method and the explicit single-shift QR method.

(c) What is the Wilkinson shift? Give heuristic reasons why this shift helps to find eigenvalues.

(d) Explain how the single-shift QR step $H - \mu I = QR$, $\overline{H} = RQ + \mu I$ where H (and, hence, $\overline{H}$) is upper Hessenberg can be carried out implicitly. That is, show how the transition from H to $\overline{H}$ can be carried out without subtracting the shift μ from the diagonal of H.

C2. Compute a QR step with the matrix

$$A = \begin{bmatrix} 2 & \epsilon \\ \epsilon & 1 \end{bmatrix}$$

(a) without shift;

(b) with shift $k = 1$.