Important: There are four categories of problems. Answer one and only one problem from each category.

A1. (a) Describe carefully the concepts of stability of a numerical method and conditioning of a mathematical problem.

(b) Describe how the two concepts interplay in solving a problem numerically.

(c) Give an example where a well-conditioned problem is solved by an unstable algorithm.

A2. The system $Ax = b$ where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 10^{-10} & 10^{-10} \\ 1 & 10^{-10} & 10^{-10} \end{bmatrix} \quad b = \begin{bmatrix} 2(1+10^{-10}) \\ -10^{-10} \\ 10^{-10} \end{bmatrix}$$

has solution $x = [10^{-10}, -1, 1]^T$.

(a) Calculate the condition number $\kappa_\infty(A)$.

(b) Show that if $(A+E)y = b$ and $\|E\|_\infty \le 10^{-8}\|A\|_\infty$, then $\|x-y\|_\infty \le 10^{-7}\|x\|_\infty$.

(c) Does small relative changes in A's entries necessarily induce large changes in x even if A is ill-conditioned? Explain what is going on.

(d) Define $D = \text{diag}\{10^{-5}, 10^5, 10^5\}$. Show $\kappa_\infty(DAD) \le 5$.

B1. (a) Let $A \in R^{m\times n}$, $u \in R^m$ and $v \in R^n$. Show the rank of the matrix $B = A - \sigma^{-1}uv^T$ is less than that of A if and only if there are vectors $x \in R^n$ and $y \in R^m$ such that $u = Ax$, $v = A^Ty$ and $\sigma = y^TAx$, in which case $\text{rank}(B) = \text{rank}(A) - 1$.

(b) Discuss how the above result can be used to break down A as a sum of rank-one matrices.

B2. (a) Show the existence of a SVD decomposition for any matrix $A \in R^{m\times n}$.

(b) Discuss at least two significant applications of SVD.

C1. (a) Describe in details what the conjugate-gradient (CG) method is.

(b) Explain where the properties of symmetry and positive-definiteness are used in the CG method?

(c) One way to solve large sparse nonsymmetric linear systems $Ax = b$, is to apply CG to the normal equations $A^TAx = A^Tb$. Discuss the pros and cons of this idea. Describe how the computation should be implemented.

C2. (a) Suppose $A = I + B$ is an $n \times n$ symmetric and positive definite matrix and $\text{rank}(B) = r$. Show that the CG method converges in at most r steps for the system $Ax = b$.

(b) Let A be a SPD matrix with the property that all of its eigenvalues are in the intervals $(1, 2)$, $[4, 5)$, and $(7, 8)$. What is the maximum number of conjugate gradient iterates required to reduce the A-norm of the error by a factor of 10^{-2}. Give the details of your analysis.

D1. When implementing Newton's method, an amateur numerical analyst makes an error in the Jacobian computation. This means that instead of $J(x)$, the Jacobian at x, our inept friend computes $J(x)+E(x)$, where E is the result of poor programming. Assume that the local convergence theory for Newton's method is applicable and that the initial iterate is near enough to the root so that an error-free iteration would converge quadratically to the root. Discuss the effect of a small, but non-zero error on the convergence. Rigorously verify your conclusions.

D2. Suppose the QR factorization $QR = A \in R^{m\times n}$ is already known. Propose an algorithm to obtain the QR factorization of

$$B := \begin{bmatrix} w^T \\ A \end{bmatrix}$$

in the most economical way.