**Important: There are three categories of problems. Answer one and only one problem from each category.**

A1. Let $A$ be an invertible real $n \times n$ matrix. Given $b \in R^n$, let $x$ be the exact solution of the linear system $Ax = b$. Assume this system is also solved by Gaussian elimination ($LU$-decomposition) with finite-precision arithmetic which gives an approximate solution $\bar{x}$. Let $r = A\bar{x} - b$. Let $C$ be an approximation to the inverse of $A$ and assume that $\delta = \|I - CA\| < 1$. Here, $\|\cdot\|$ denotes a vector norm on $R^n$ and the associated induced matrix norm.

(a) Prove that

$$\frac{\|Cr\|}{1+\delta} \leq \|x - \bar{x}\| \leq \frac{\|Cr\|}{1-\delta}.$$

(b) Show that if the factors of the $LU$-decomposition of $A$ are available, then computing $Cr$ requires $O(n^2)$ multiplications.

A2. Consider the following situation:

(a) Find the $LU$-decomposition of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \\ 1 & 16 & 81 & 256 \end{bmatrix}.$$

and use this decomposition to solve the system of equation $Ax = b$ with $b = [3, 1, -15, -107]^T$.

(b) Suppose we want to solve a linear system of equations, but after having computed the $LU$-decomposition, we discover that one column in the original matrix is wrong. How can the decomposition nevertheless be used to find the correct solution? Formulate a corresponding algorithm.

(c) Apply your algorithm to the above linear system where the first column of $A$ is to be replaced by $[0, 0, 6, 36]^T$.

B1. Given $A \in R^{m \times n}$, let $A = U\Sigma V^T$ denote the singular value decomposition of $A$. Then $A^+ := V\tilde{\Sigma}U^T$ is known as the pseudo-inverse of $A$ where $\tilde{\Sigma} := [\tau_\mu \delta_{\mu\nu}] \in R^{n \times m}$ and

$$\tau_\mu := \begin{cases} \sigma_\mu^{-1}, & \text{if } \sigma_\mu \neq 0 \\ 0, & \text{if } \sigma_\mu = 0 \end{cases}$$

is known as the pseudo-inverse of $A$.

(a) Compute a singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}.$$

(b) Let $A = [a_{11}, a_{12}] \in R^{1 \times 2}$. Compute $A^+$.

(c) Let $A \in R^{m \times n}$. Show:

$$A^+ = (A^T A)^+ A^T = A^T (AA^T)^+.$$

B2. Suppose $y$ is defined by the formula

$$y = \sqrt{z + \sqrt{z + \sqrt{z + \ldots}}}$$

for $z \in R_+$.

(a) Find an iterative method for computing the number $y$.

(b) Determine for which choices of the starting values the iteration will converge. Justify your answers.

(c) Compute $y$.

B3. Let $A$ be an $n \times n$ symmetric and positive definite matrix whose eigenvalues all lie in the sets $(.1, .2)$, $(2.9, 3.1)$, and $(99.9, 100)$. Give an estimate on the number of conjugate gradient iterations $k$ needed so that

$$\|x_k - A^{-1}b\| \leq .1\|x_0 - A^{-1}b\|.$$

Justify your estimate.

C1. A matrix with the property that the sums of the absolute values of the entries in each row are equal is called *row-equilibrated*.

(a) Show that every nonsingular matrix can be transformed into a row-equilibrated matrix by pre-multiplying by a diagonal matrix.

(b) Show that if $A$ is a row-equilibrated matrix, then

$$\text{cond}_\infty(A) \leq \text{cond}_\infty(DA)$$

for every nonsingular diagonal matrix $D$; i.e., equilibration improved the condition of the matrix.

(c) Let $A(r)$ be the $1 \times 1$ matrix $\{r\}$. For $r \neq 0$ let $k(r)$ denote the condition number of $A(r)$ (in any norm). Compute $dk(r)/dr$ for $r \neq 0$.

C2. Assume $A$ is a real symmetric $N \times N$ matrix with eigenvectors $x_1, x_2, ..., x_N$ with corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ and that the eigenvalues satisfy

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_N|.$$

Starting with an initial guess $q^{(0)}$ for which $\|q^{(0)}\|_2 = 1$, we define the sequence $\{q^{(j)}\}_{j\geq 0}$ by

$$q^{(j+1)} = \frac{1}{C_{j+1}} A q^{(j)}$$

where the scale factors $C_j$ are chosen so that $\|q^{(j)}\|_2 = 1$.

(a) Prove that the vectors $q^{(j)}$ converge as $j \to \infty$ to a certain multiple of the eigenvector $x_1$ of the matrix $A$.

(b) Suppose we also define a sequence of scalars $\{\mu_j\}_{j\geq 0}$ by the Raleigh quotient

$$\mu_j = \frac{q^{(j)T} A q^{(j)}}{q^{(j)T} q^{(j)}}.$$

Show that the scalars $\mu_j$ converge as $j \to \infty$ to the eigenvalue $\lambda_1$. Also show that the rate of convergence of the scalars $\mu_j$ to the eigenvalue $\lambda_1$ is in general twice as fast as the convergence of the vectors $q^{(j)}$ to a multiple of $x_1$.

C3. Let $A$ be an $n \times n$ symmetric and positive definite matrix with exactly $m$ distinct eigenvalues. ($n \geq m$, obviously!) Show that the conjugate gradient method will find a solution to $Ax = b$ in at most $m$ iterations.