Possible Questions for Part A Qualifying Examinations in MA 529
Numerical Analysis, August 1994

Directions: If you are taking this exam under the old system do any three problems. If you and taking this exam under the new system. Do three problems from the 529-530 exams and at least one from each.

1. (a) State the standard assumptions for systems of nonlinear equations (*ie* the assumptions required for Newton's method to converge locally q-quadratically to a solution).
   (b) Define linear convergence, superlinear convergence, and quadratic convergence for a sequence.
   (c) Show that if $K > 0$ and e_0, e_{-1} are sufficiently small that the sequence given by
   $$e_{n+1} = K e_n e_{n-1}$$
   converges q-superlinearly with q-order $(\sqrt{5}+1)/2$.
   (d) State and prove the theorem on local linear convergence for the chord method for nonlinear equations.

2. (a) What is the condition number of a matrix?
   (b) Compute the l^1, l^∞, and l^2 condition numbers of
   $$\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$$
   (c) Let $Ax = b$ and let y be the solution of $(A+E)y = b+e$. State and prove the result from class that relates $\|x-y\|/\|x\|$ to $\|e\|/\|b\|$, $\|E\|/\|A\|$ and the condition number of A.

3. In the questions that follow, "describe" means to give a pseudo-code representation or detailed outline of an algorithm.
   (a) Describe the GMRES algorithm for solution of linear systems.
   (b) What is the minimization principle for the GMRES?
   (c) Using the minimization property, prove that the GMRES iteration for solution of nonsingular linear systems will find a solution in finitely many iterations.
   (d) Let A be a normal matrix whose eigenvalues are in the intervals $(-1, -.9)$ and $(99.9, 100)$. Give an estimate for the number of GMRES iterates for the linear equation $Ax = b$ required to reduce the relative residual by a factor of 10^{-3}.

4. (a) What is the QR factorization of an $M \times N$ matrix?
   (b) Prove the QR factorization is unique.
   (c) Describe an algorithm to compute the QR factorization.
   (d) How is the QR factorization used to solve linear least squares problems?
   (e) What are the normal equations for a least squares problem? How does the cost of solving the normal equations compare with using a QR factorization? What are some advantages of using a QR factorization?

5. (a) What does it mean for a matrix, A, to be positive definite?
   (b) Show that if A is symmetric and positive definite (spd) that $a_{ii} > 0$.
   (c) What is the Cholesky factorization of a spd matrix? Prove that it is unique.
   (d) Describe an algorithm for computing the Cholesky factorization of a spd matrix.
   (e) How many floating point operations are required to form the Cholesky factorization?

6. (a) State the minimization property for conjugate gradient iteration.
   (b) Using the minimization property, prove that the conjugate gradient iteration for solution of symmetric positive definite linear systems will find a solution in finitely many iterations.
   (c) What is the CGNR algorithm for nonsymmetric linear systems?

(d) Formulate a minimization principle for CGNR.

(e) Suppose A is nonsingular and all the singular values of A lie in the intervals $[.9, 1.2]$ and $[99.9, 100.]$. How many iterations will the CGNR algorithm require to reduce the norm of the residual by a factor of 10^{-3}?

7. Suppose you try to find an eigenvalue/eigenvector pair of an $N \times N$ matrix A by solving the system of $N+1$ **nonlinear** equations

$$F(\phi, \lambda) = \begin{pmatrix} A\phi - \lambda\phi \\ \phi^T\phi - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for the vector $(x, \lambda)^T \in R^{N+1}$. What is the Jacobian of this system? If $(\phi, \lambda) \in R^{N+1}$ is an eigenvector-eigenvalue pair, when is $F'(\phi, \lambda)$ nonsingular?