MA 529 Preliminary Exam. January, 1994

Directions: Do one of problems 1 or 2, and do two of 3, 4, or 5.

1. (a) Define the condition number of a matrix.

   (b) If A is a nonsingular $N \times N$ matrix, $Ax = b$, and $(A + \Delta A)(x + \Delta x) = b$, estimate Δx in terms of A, ΔA, x, b, and the condition number of A. Give a complete derivation. Do not simply quote a result.

   (c) Compute the l^1 condition number of

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 10 \end{pmatrix}.$$

2. (a) What is the Cholesky Factorization and what is it for?

   (b) Using the definitions, show that a strictly diagonally dominant symmetric tridiagonal matrix with positive diagonal is symmetric and positive definite.

   (c) Derive an operation count of the form $N^p + O(N^{p-1})$ for the Cholesky factorization of a spd tridiagonal matrix.

3. (a) Derive a count for the number of scalar products and matrix-vector products in a single conjugate gradient iteration.

   (b) What are the minimization principles for CG and GMRES?

   (c) Using the minimization principle for CG iteration, show that CG will solve an $N \times N$ spd linear system of equations in at most N steps.

   (d) Assume that A is spd and that

$$\sigma(A) \subset (1, 1.1) \cup (99, 100).$$

   Give upper estimates based for the number of CG iterations required to reduce the A norm of the error by a factor of 10^{-3} and for the number of CG iterations required to reduce the A-norm of the error by a factor of 10^{-3}. Assume that $x_0 = 0$.

4. (a) State the standard assumptions for systems of nonlinear equations (*i. e.*the assumptions required for Newton's method to converge locally q-quadratically to a solution).

   (b) Suppose that one solves the linear equation for the Newton step by an iterative method which terminates on small residuals. So the iteration would look like

$$x_{n+1} = x_n + s_n$$

   where the computed step almost satisfies the equation for the true Newton step in that

$$\|F'(x_n)s_n + F(x_n)\| \le \eta_n \|F(x_n)\|.$$

   for some small η_n. Assume that the standard assumptions hold. Show that if $\bar{\eta}$ is sufficiently small, $\eta_n \le \bar{\eta}$, then the iteration converges locally q-superlinearly to the solution x^*.

5. (a) What are the power method, the inverse power method, and the QR algorithm for computing eigenvalues? You need not give the technical implementation details, just give the basic iteration and describe the goal of each algorithm.

   (b) Let A be an spd $N \times N$ matrix with distinct eigenvalues. Prove that the sequence of approximate eigenvalues computed by the power method converges to the largest eigenvalue of A.

   (c) Let S be a nonsingular matrix. Show that $S^{-1}AS$ has the same eigenvalues as A.