Numerical Analysis, August 1995

**Directions:** This examination is two pages long. The questions for MA 529 are on page 1 and those for MA 530 on page 2. Do three problems, taking least one from each of 529 and 530.

MA 529

1. (a) Define CG, GMRES, CGNR, and CGNE iterations in terms of the problems they are designed to solve and the Krylov subspace minimization principles that they satisfy.
   (b) Let $K$ be a diagonalizable $N \times N$ matrix with rank 3 and let $A = I - K$ be nonsingular. How many iterations will be needed to solve $Ax = b$ with GMRES? How about CGNR? Rigorously justify your answers.
   (c) Show that if $A$ has constant diagonal then PCG with Jacobi preconditioning produces the same iterates as CG with no preconditioning.
   (d) Assume that $A$ is spd and that
   $$\sigma(A) \subset (.95, 1.05) \cup (1.9, 2.1).$$
   Give upper estimates based for the number of CG iterations required to reduce the $A$ norm of the error by a factor of $10^{-3}$ and for the number of CG iterations required to reduce the residual by a factor of $10^{-3}$.
2. (a) What does it mean for a function to be Lipschitz continuous?
   (b) What are the standard assumptions made for systems of nonlinear equations?
   (c) Give an example of a sequence that is r-superlinearly but not q-linearly convergent.
   (d) Assume that the standard assumptions hold. Suppose that $A(x)$ is a matrix-valued function of $x$ such that
   $$\|I - A(x)F'(x)\| \leq \|e\|^{1/2}$$
   for all $x$ sufficiently near $x^*$. Show that if $x_c$ is sufficiently near $x^*$ and
   $$x_+ = x_c - A(x_c)F(x_c)$$
   that there is $K$ such that
   $$\|e_+\| \leq K\|e_c\|^{3/2}$$
3. (a) What is the QR and SVD decompositions of an $M \times N$ matrix?
   (b) Show that the QR decomposition is unique if $R$ is required to have positive diagonal.
   (c) Describe a stable algorithm to compute the QR decomposition.
   (d) How are the QR and SVD decompositions used to solve linear least squares problems?
   (e) What are the normal equations for a least squares problem? How does the cost of solving the normal equations compare with using a QR factorization? What are some advantages of using a QR factorization?
4. (a) What are the power method, the inverse power method, and the QR algorithm for computing eigenvalues? You need not give the technical implementation details, just give the basic iteration and describe the goal of each algorithm.
   (b) Let $A$ be an spd $N \times N$ matrix with distinct eigenvalues. Prove that the sequence of approximate eigenvalues computed by the power method converges r-linearly to the largest eigenvalue of $A$.
   (c) What is Rayleigh quotient iteration? Why is it useful?

1. Suppose a complicated function $f(x)$ is to be approximated by an easier function of the form $\phi(x; a_0, \ldots, a_n)$ where $a_o, \ldots, a_n$ are parameters to be determined so as to characterize the best approximation of $f$.
   (a) Define the sense in which the approximation is realized when we say
      i. The function $\phi$ interpolates $f$.
      ii. The function $\phi$ is a least-square approximation to $f$.
      iii. The function $\phi$ is a min-max approximation to $f$.
   (b) Briefly discuss the advantages and disadvantages of each type of approximations.
   (c) Suppose $\phi(x; a_0, \ldots, a_n)$ depends linearly on the parameters $a_i$, i.e., $\phi(x; a_0, \ldots a_n) = a_0\phi_0(x) + \ldots + a_n\phi_n(x)$, where $\phi_i(x)$ are given and fixed basis functions. Explain how the least squares approximation problem can be solved.
   (d) No technical details are needed, but describe how a min-max approximation can be found.
2. Suppose that $f(x)$ is defined on the interval $[0, 2\pi]$.
   (a) What is meant by the Fourier series $\mathcal{F}_f(x)$ of $f(x)$ on the interval $[0, 2\pi]$? Define the Fourier coefficients of $f(x)$. State a theorem on the convergence of Fourier series.
   (b) What is meant by a trigonometric interpolation $\mathcal{T}_f(x)$ at the support points $(x_k, f_k)$, $k = 0, 1, \ldots, 2m-1$, where $x_k = \frac{2\pi k}{2m}$ and $f_k = f(x_k)$? Describe a close-form solution of the coefficients involved in $\mathcal{T}_f(x)$.
   (c) How are the Fourier series $\mathcal{F}_f$ and the trigonometric interpolation $\mathcal{T}f$ of $f(x)$ related?
   (d) Outline the basic idea behind Fast Fourier Transform (FFT). It suffices to illustrate the idea by an example.
   (e) Suppose $p(x) = \beta_0 + \beta_1 e^{ix} + \ldots + \beta_{2m-1}e^{i(2m-1)x}$ is the phase polynomial that interpolates the support points $(x_k, f_k)$, $k = 0, 1, \ldots, 2m-1$. Show that the truncated $s$-segment $p_s(x) = \beta_0 + \beta_1 e^{ix} + \ldots + \beta_s e^{isx}$ minimizes the sum

$$S(q) := \sum_{k=0}^{2m-1} |f_k - q(x_k)|^2$$

   over all phase polynomials $q(x) = \gamma_0 + \gamma_1 e^{ix} + \ldots + \gamma_s e^{isx}$.
3. Let $f(x)$ be a smooth function defined over the interval $[-1, 1]$,
   (a) Describe how the Hermite polynomial $H(x)$ that interpolates $f(x_i)$ and $f'(x_i)$ for $i = 1, \ldots n$ can be formulated.
   (b) Let the integral of $H(x)$ over $[-1, 1]$ be denoted as $Q(f) := \sum_{i=1}^{n} (\alpha_i f(x_i) + \beta_i f'(x_i))$. Show that $Q(f)$ is a quadrature for $f$, i.e., $\beta_i = 0$ for all $i = 1, \ldots, n$, with degree of precision $2n-1$ if and only if the nodes $x_i$, $i = 1, \ldots, n$, are the roots of the $n$-th Legendre polynomial,
   (c) Show that the weights $\alpha_i$ are always positive and that

$$\alpha_i = \int_{-1}^{1} \ell_i(x)dx$$

   where

$$\ell_i(x) = \prod_{k=1, k\neq i}^{n} \frac{x - x_k}{x_i - x_k}.$$

   (d) If $p$ and $q$ are polynomials of degree less than $n$, show that $\int_{-1}^{1} f(x)g(x)dx = \sum_{i=1}^{n} \alpha_i f(x_i)g(x_i)$.