Direction: There are two categories of problems. Do three problems, taking at least one from each category.

A1. (a) Define the condition number of a matrix.

(b) Let $Ax = b$ and let y be the solution of $(A + E)y = b + e$. State and prove the well-known inequality that relates $\|x - y\|/\|x\|$ to $\|e\|/\|b\|$, $\|E\|/\|A\|$ and the condition number of A.

(c) Suppose A is symmetric positive definite and the l^2 condition number $\kappa_2(A)$ of A satisfies $.9 \leq \kappa_2(A) \leq 1.1$. Give an estimate for the number of CG iterations it will take to reduce the A-norm of the error by a factor of 10.

A2. (a) State and prove the theorem on local quadratic convergence of Newton's method.

(b) Compute (by hand) the Newton sequence for the equation $x^2 = 0$ with initial iterate $x_0 \neq 0$. Does this sequence converge quadratically? Is the convergence better or worse? Relate your conclusion to the theorem.

(c) Compute the first Newton iterate for the equation $x^3 + x = 0$ with initial iterate $x_0 \neq 0$. From that derive the q-order of convergence for x_0 sufficiently near 0. Does this sequence converge quadratically? Is the convergence better or worse? Relate your conclusion to the theorem.

A3. Consider the iterative method

$$C\frac{v^{(n+1)} - v^{(n)}}{\Delta t} + Av^{(n)} = b, n = 0, 1, \ldots$$
$$v^{(0)} = v_0,$$

where A and $C \in R^{n \times n}$ are symmetric positive matrices, b and $v_0 \in R^n$ are given vectors, and Δt is a positive parameter. Note that this corresponds to the Euler forward scheme applied to the ODE

$$C\frac{dv(t)}{dt} + Av(t) = b, v(0) = v_0.$$

(a) Give a condition ensuring the convergence of the sequence $\{v^{(n)}\}$.

(b) For what C and Δt do the above $v^{(n)}$'s correspond to the iterates of the Jacobi iterative method applied to the problem $Au = b$? Discuss the meaning of this relationship.

(c) Assume $C = I_n$. Give explicitly, in terms of the eigenvalues of A only, a condition on the parameter Δt ensuring convergence of $\{v^{(n)}\}$ for all $v_0 \in R^n$.

(d) Discuss whether or not methods like (P)CG or GMRES also admit a "temporal interpretation" of this kind?

B1. Consider a rational function $R_{mk}(x)$ of the form

$$R_{mk}(x) = \frac{P_m(x)}{Q_k(x)}$$

where $P_m(x)$ and $Q_k(x)$ are polynomials of degrees, m and k, respectively. Padé approximation of a function $f(x)$ is a special $R_{mk}(x)$ such that its derivatives agree with as many as possible those of $f(x)$ at $x = 0$.

(a) What is the highest possible order derivative that $f(x)$ and $R_{mk}(x)$ can match? Justify why this order can be achieved.

(b) Calculate the Padé approximation $R_{22}(x)$ of the exponential function e^x over the interval $[-1, 1]$.

(c) For any $x \in (-\infty\infty)$, write

$$x \log_2(e) = X + F$$

where X is an integer and F is a fraction such that $-1 < F < 1$. Explain how e^x can be calculated with information of X and F.

B2. It is known that the integral

$$\int_0^1 \frac{2}{2 + \sin 10\pi x} dx = \frac{2}{\sqrt{3}} \approx 1.1547005383792515290182975610039\dot{1}4.$$

(a) Describe two different numerical procedures that can evaluate the integral with accuracy up to the eighth digit. The description should be specific on all parameters involved so as to guarantee the accuracy desired.

(b) Explain what is meant by a Gaussian quadrature formula.

B3. Suppose we want to apply the midpoint rule

$$y_{n+1} = y_{n-1} + 2hf_n$$

to the initial value problem

$$y' = -y, \ y(0) = 1.$$

(a) Let the starting values be $y_0 = 1$ and $y_1 = -h + \sqrt{h^2 + 1} + \epsilon$ where ϵ is the machine precision. What values y_i are to be expected for arbitrary step size h?

(b) Use the above observation to explain a general notion of numerical instability in numerical ODE for initial value problems. Briefly discuss how such a phenomenon can be avoided when designing a new numerical scheme.