Direction: There are two categories of problems. Do three problems, taking at least one from each category.

A1. Concerning the Gaussian elimination method for solving $Ax = b$,

(a) Prove that if A is a positive definite matrix, then after a step of Gaussian elimination the reduced matrix A_1 in $A \to \begin{pmatrix} a_{11} & \star \\ 0 & A_1 \end{pmatrix}$ must be positive definite.

(b) Prove that if A is strictly column diagonally dominant, i.e., for each k,

$$|a_{kk}| > \sum_{j \neq k} |a_{jk}|,$$

then no row interchanges need to take place.

A2. Suppose A is a symmetric positive definite $N \times N$ matrix and $b \in R^N$.

(a) Let $\{x_n\}$ and $\{y_n\}$ be the CG and the GMRES iterates, respectively, for the solution of $Ax = b$ with $x_0 = 0$. Is $x_n = y_n$ for all n? Why or why not?

(b) Give conditions on the eigenvalues of A that guarantee that the iteration

$$x_{n+1} = x_n - (b - Ax_n)/3$$

will converge to $A^{-1}b$.

(c) Show that the the iteration

$$x_{n+1} = x_n - h(b - Ax_{n+1})$$

converges for all $h > 0$ and compute the limit.

A3. Concerning the eigenvalue computation,

(a) Compute one QR step in the QR algorithm for the matrix

$$A = \begin{bmatrix} 2 & \epsilon \\ \epsilon & 1 \end{bmatrix}$$

with shift constant $\mu = 1$ and without shift at all. Comparing the two resulting isospectral matrices, the one with shift $\mu = 1$ is closer to a diagonal matrix. Explain why?

(b) Let A be a real $n \times n$ symmetric matrix. For any nonzero $x \in R^n$, the Rayleigh quotient $\rho(x)$ of x is defined to be

$$\rho(x) := \frac{\langle Ax, x \rangle}{\langle x, x \rangle}$$

where $\langle \cdot, \cdot \rangle$ denote the usual Euclidean inner product in R^n. It is easy to see that the Rayleigh quotient evaluated at an eigenvector is exactly the corresponding eigenvalue. Show that, however, if a given x is an $O(\epsilon)$ approximation to an eigenvector, the $\rho(x)$ is an $O(\epsilon^2)$ approximation to the corresponding eigenvalue.

B1. Consider the numerical integration of a smooth function $f(x)$ over a finite interval $[a, b]$.

(a) Describe how a general Newton-Cotes quadrature can be formulated from the n-th degree Lagrange interpolating polynomial P_n of f.

(b) For $n = 1$ and 2, describe explicitly the Newton-Cotes quadratures. What is the respective degree of precision of these formulas?

(c) Discuss the fundamental difference between the Newton-Cotes quadrature and the Gaussian quadrature.

B2. Concerning the Fast Fourier Transform,

(a) Suppose entries of $x = [x_0, x_1, \ldots, x_{N-1}]^T$ represent discrete samples of a continuous function $f(t)$ at points $t_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. What is the connection between the trigonometric interpolation of f at (t_k, x_k), $k = 0, 1, \ldots, N-1$ and the discrete Fourier transform $y = F_N x$ where $F_N = [f_{pq}]$ is the Frobenius matrix with

$$f_{pq} := \omega_N^{pq}$$
$$\omega_N := e^{-2\pi i/N}$$

and $i = \sqrt{-1}$?

(b) Use the example $N = 16$ to explain how the radix-2 splitting idea can be employed to speed up the calculation of the product $y = F_{16}x$.

B3. Concerning the Runge-Kutta methods for ordinary differential equations,

(a) From a Butcher array

$$\begin{array}{c|c} a & B \\ \hline & c^T \end{array}$$

where

$$a = [a_1, a_2, \ldots, a_s]^T$$
$$c = [c_1, c_2, \ldots, c_s]^T$$
$$B = [b_{ij}] \in R^{r\times r}$$

describe how a general s-stage Runge-Kutta method is complete specified.

(b) Find conditions for a generic 2-stage explicit Runge-Kutta method to be of order two. Determine all such methods.

(c) What can you say about the stability of these methods?