Preliminary Examination in Numerical Analysis, January 2000

Direction: There are three categories of problems. Answer one and only one problem from each category.

Category A

A1. 1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Solve the least squares problem

$$\min_{x \in R^2} \| Ax - b \|_2.$$

2. Let $A_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$, $\epsilon > 0$ and b as above. Solve the least squares problem

$$\min_{x \in R^2} \| A_\epsilon x - b \|_2.$$

What happens when $\epsilon \to 0$? Compared with the previous problem, does your observation contradict any result seen before? Explain.

3. Explain briefly how to solve least square problems using the SVD. How would you deal numerically with the above "discontinuity", assuming for instance ϵ results from round-off errors?

4. Let $B = U\Sigma V^T$ denote the reduced SVD of a $m \times n$ matrix B. Suppose $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$. Define $B_k = U diag(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)V^T$ for $k \leq n$. What is the rank of B_k? Show that $\| B - B_k \|_2 = \sigma_{k+1}$.

A2. The following questions concern the eigenvalue computation:

1. The matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1.5 & 1 & 1.5 \\ 2 & 0.5 & 1.5 \end{bmatrix}$$

has eigenvalue $\lambda = 4$ with corresponding eigenvector $x = [1, 1, 1]^T$. Construct a Householder matrix H such that

$$HAH^T = \begin{bmatrix} 4 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

and determine the remaining eigenvalues.

2. Show that the Hessenberg form of a matrix is preserved by the QR algorithm.

3. Assume that B is a real symmetric $n \times n$ matrix with eigenvalue λ of multiplicity $n-1$ and a further eigenvalue $\mu \neq \lambda$. Show that $B = \lambda I + (\mu - \lambda)xx^T$, where $x^T x = 1$ and that by at most $n-1$ Jacobi transformations B becomes diagonal. (*Hint: Jacobi transformation refers to an orthogonal similarity transformation by plan rotations.*)

Category B

B1. The Laguerre polynomials $\{L_n(x)\}_{n=0}^{\infty}$ are orthogonal with the inner product

$$(f,g) = \int_0^{\infty} e^{-x} f(x) g(x)\, dx.$$

1. Given $L_0(x) \equiv 1$, find $L_1(x)$ and $L_2(x)$ (hint; $\int_0^{\infty} e^{-x} x^n\, dx = n!$) .
2. Let f be a function defined on $[0,\infty)$. We seek an approximation of f of the form $p_n(x) = \sum_{k=0}^{n} a_k L_n(x)$ such that
$$\int_0^{\infty} e^{-x}(f(x) - p_n(x))^2\, dx$$
is minimized. Describe the corresponding normal equations and solve them, i.e., give the expression of the coefficients a_k, $k = 0,\ldots,n$. (The function f is supposed to be such that all the above integrals make sense).
3. We want to approximate $\int_0^{\infty} e^{-x} f(x)\, dx$ using Gaussian quadrature. We ask for the cases $f(x) = 1, x, x^2$ and x^3 to be integrated exactly. Set up the equations for the nodes and weights. Solve those equations. Hint: the nodes are closely related to the roots of one of the $L_n(x)$'s.
4. Comment on the numerical merits, or the lack thereof, of the two previous approximations methods.

B2. The following questions concern iterative methods for nonlinear systems:

1. Show that
$$x_{n+1} := \frac{x_n(x_n^2 + 3a)}{3x_n^2 + a}, \quad n = 0, 1, \ldots$$
is a method of order three for computing the square root of a positive number a.
2. Show that for a nonsingular $n \times n$ matrix A the sequence
$$A_{n+1} := A_n[2I - AA_n], \quad n = 0, 1, \ldots$$
converges quadratically to the inverse A^{-1}, provided that $\|I - AA_0\| < 1$.
3. The eigenvalue problem $Ax = \lambda x$ for an $n \times n$ matrix A is equivalent to the equation $f(z) = 0$, where $f : R^n \times R \longrightarrow R^n \times R$ is defined by
$$f : \begin{pmatrix} x \\ \lambda \end{pmatrix} \longrightarrow \begin{pmatrix} Ax - \lambda x \\ x^T x - 1 \end{pmatrix}.$$
Write down Newton's method for this equation.

Category C

C1.

1. Give the description of a general Runge-Kutta method.
2. Give the description of a general explicit Runge-Kutta method.
3. What is meant by A-stability?
4. Are there any explicit A-stable explicit Runge-Kutta method? If yes, give one, if no, explain.
5. Give an example of an A-stable method.

C2. Let $u_1, \ldots, u_n \in C[a,b]$ be linearly independent and let $x_1, \ldots, x_n \in [a,b]$ be distinct. For given values $y_1, \ldots, y_n \in R$, consider the interpolation problem of finding a function $u \in \mathcal{U}_n := \operatorname{span}\{u_1, \ldots, u_n\}$ with the property

$$u(x_j) = y_j, \quad j = 1, \ldots n.$$

1. Show that the following three properties are equivalent:
   (a) The interpolation problem is uniquely solvable for each given set of values $y_1, \ldots, y_n \in R$.
   (b) Each function $u \in \mathcal{U}_n$ with zeros $u(x_j) = 0$ for $j = 1, \ldots, n$ vanishes identically.
   (c) The $n \times n$ matrix with entries $u_k(x_j)$ for $j, k = 1, \ldots, n$ is nonsingular.
2. Excluding the standard basis of polynomials $1, x, x^2, \ldots$, give two distinct sets $\{u_1, \ldots, u_n\}$ of functions that have the above properties. Justify your answer.