Preliminary Examination in Numerical Analysis (580-780), August 2001

Direction: There are three categories of problems. Answer one and only one problem from each category.

Category A

A1. Consider the task of solving the linear equation $A\mathbf{x} = \mathbf{b}$ by linear iterative schemes.

1. Suppose that $A \in \mathbb{R}^{n\times n}$ is split as $A = (1+\omega)S - (T + \omega S)$ whereas $S^{-1}T$ is known *a priori* to have real eigenvalues $1 > \lambda_1 \geq \lambda_2 \geq \lambda_n$. Find the values of $\omega \in \mathbb{R}$ for which the iterations from
$$(1+\omega)S\mathbf{x}^{(k+1)} = (T+\omega S)\mathbf{x}^{(k)} + \mathbf{b} \tag{1}$$
converge for all starting values $\mathbf{x}^{(0)}$ to $\mathbf{x}$. Determine also the value ω for which the convergence rate is maximum.
2. Suppose that the original linear equation is written in block form
$$\begin{bmatrix} A_1 & B \\ C & A_2 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}, \tag{2}$$
where A_1 and A_2 are invertible. Consider the two methods
$$\begin{cases} A_1\mathbf{y}^{(k+1)} + B\mathbf{z}^{(k)} = \mathbf{b}_1 \\ C\mathbf{y}^{(k)} + A_2\mathbf{z}^{(k+1)} = \mathbf{b}_2; \end{cases} \quad \text{and} \quad \begin{cases} A_1\mathbf{y}^{(k+1)} + B\mathbf{z}^{(k)} = \mathbf{b}_1 \\ C\mathbf{y}^{(k+1)} + A_2\mathbf{z}^{(k+1)} = \mathbf{b}_2. \end{cases} \tag{3}$$
Describe sufficient conditions in order for the two schemes to be convergent for any choice of the initial data $\mathbf{y}^{(0)}$ and $\mathbf{z}^{(0)}$.
3. Consider the 2-dimensional linear system where
$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \tag{4}$$
and the associated objective function
$$\Phi(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T A\mathbf{y} - \mathbf{y}^T\mathbf{b}. \tag{5}$$
Perform some iterations of the gradient method and the conjugate gradient method. Give a graphical interpretation of the iterates.

A2. The equation of motion arising in many mechanics applications appears as a linear second-order differential equations:
$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = f(\mathbf{x}), \tag{6}$$
where $\mathbf{x} \in \mathbb{R}^n$ and M, C, $K \in \mathbb{R}^{n\times n}$. Usually M is symmetric and positive definite. It is known that the general solution to the homogeneous equation is of vital preliminary to the stability of the subsequent dynamical behavior.

1. Show that by proposing a homogeneous solution of the form
$$\mathbf{x}(t) = \mathbf{v}e^{\mu t},$$
both $\mathbf{v}$ and μ are solutions to the quadratic eigenvalue problem
$$(\mu^2 M + \mu C + K)\mathbf{v} = 0. \tag{7}$$

2. Generally, how many eigenvalues are expected? Explain why?
3. Show that (7) can be reduced to a $2n \times 2n$ standard eigenvalue problem

$$A\mathbf{x} = \mu \mathbf{x} \tag{8}$$

as well as a $2n \times 2n$ generalized eigenvalue problem

$$B\mathbf{z} = \mu C\mathbf{z}. \tag{9}$$

Be specific about the matrices A, B, C and the relationship among $\mathbf{v}, \mathbf{x}, \mathbf{z}$.
4. Assume that in (9) both matrices B and C are symmetric and that C is positive definite (such a system is called a *symmetric definite pencil.*) Show that in this case, all eigenvalues are real and the corresponding eigenvectors are linear independent.
5. Assuming that the shift s is an approximate eigenvalue for the matrix pencil (B, C) (not necessarily symmetric definite), discuss the convergence behavior of the inverse power method
   (a) Choose an initial vector,
   (b) For $k = 1, 2, \ldots$, do until convergence

$$\begin{cases} \text{Solve } (B - sC)\mathbf{y}^{(k)} = C\mathbf{x}^{(k-1)}, \\ x^{(k)} = \frac{\mathbf{y}^{(k)}}{\|\mathbf{y}^{(k)}]\|}. \end{cases} \tag{10}$$

Category B

B1. Consider numerical methods for solving the Fredholm integral equation of the second kind, i.e.,

$$\varphi(x) - \int_a^b K(x,y)\varphi(y)dy = f(x), \quad x \in [a,b] \tag{11}$$

with continuous kernel K and $\varphi \in C[a,b]$.

1. (The Nyström method) Suppose that $\varphi(x)$ is represented by a set of discrete data $\varphi_n := [\varphi(x_0), \ldots, \varphi(x_n)]^T$ at quadrature points, i.e., abscissas, $x_0, \ldots, x_n \in [a,b]$. Describe how the solution to the integral equation (11) is approximated by the solution of a linear system

$$\varphi_n - A_n \varphi_n = f_n. \tag{12}$$

Be specific about entries in the matrix A_n.
2. (The collocation method) Suppose that $\varphi(x)$ is approximated by the Lagrange interpolation polynomial

$$\varphi_n(x) = \sum_{k=0}^{n} \gamma_k \ell_k(x) \tag{13}$$

where $\ell_k(x)$, $k = 0, \ldots, n$, are the Lagrange factors

$$\ell_k(x) = \prod_{i=1, i \neq k}^{n} \frac{x - x_i}{x_k - x_i} \tag{14}$$

at a predesignated set of $n+1$ points $x_0, \ldots, x_n \in [a,b]$ (called the collocation points.) It is desired that the approximate solution $\varphi_n(x)$ satisfies (11) at all points $x_0, \ldots, x_n \in [a,b]$. Describe how the values γ_k should be determined from a linear system.

3. In the collocation method, the basis needs not be the Lagrange factors. Suppose that the linear splines are used over a equally spaced grid with $h = (b-a)/n$ and $x_j = a + jh$, i.e., suppose that

$$\ell_k(x) = \begin{cases} \frac{1}{h}(x - x_{k-1}), & x \in [x_{k-1}, x_k], \\ \frac{1}{h}(x_{k+1} - x), & x \in [x_k, x_{k+1}], \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Describe how the approximate solution $\varphi_n(x)$ should be found.

4. Compare the pros and cons of the Nyström method with the collocation method.

B2. With respect to a weight function $\omega(x) \geq 0$ over the interval $[-1, 1]$, define the inner product of two functions by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\omega(x)dx \tag{16}$$

and the norm $\|f\| = \langle f, f \rangle^{1/2}$, whenever the integrals exist. With respect to this inner product, a sequence of orthogonal polynomials $\{p_k(x)\}$, $k = 0, 1, \ldots$, can be defined, i.e., $p_k(x)$ is a polynomial of degree precisely k and satisfies $\langle p_k, p_m \rangle = 0$ if $m \neq k$. For any function $f : [-1, 1] \to \mathbb{R}$ satisfying $\|f\| < \infty$, the series

$$\mathcal{S}(f) = \sum_{k=0}^{\infty} \hat{f}_k p_k, \tag{17}$$

with coefficients

$$\hat{f}_k = \frac{\langle f, p_k \rangle}{\langle p_k, p_k \rangle}, \tag{18}$$

is called the *generalized Fourier series* of f. For any positive integer n, define the polynomial

$$f_n(x) = \sum_{k=0}^{n} \hat{f}_k p_k(x). \tag{19}$$

1. Show that f_n is the polynomial that minimizes $\|f - q\|$ among all polynomials of degree less than or equal to n.
2. Suppose that the coefficient $\hat{f}_k$ is approximated by the value $\tilde{f}_k$ obtained by approximating the integrals involved in (18) by some quadrature rules. How is this new polynomial

$$f_n^*(x) = \sum_{k=0}^{n} \tilde{f}_k p_k(x) \tag{20}$$

related to the original f?

3. Show that if $p_{-1}(x) = 0$ and $p_1(x) = 1$, then the orthogonal polynomials satisfies a three-term recursive formula

$$p_{k+1}(x) = (x - \alpha_k)p_k(x) - \beta_k p_{k-1}(x), \quad k \geq 0. \tag{21}$$

Express α_k and β_k in terms of p_k and p_{k-1}.

4. Show that all k roots of $p_k(x)$ are real, distinct, and in the interval $(-1, 1)$.
5. Show that the Chebyshev polynomials

$$T_k(x) = \cos(k \arccos x), \quad , k = 0, 1, 2, \ldots \tag{22}$$

are orthogonal with respect to the weight function $\omega(x) = (1 - x^2)^{-1/2}$ over $(-1, 1)$ and satisfy the recursive formula

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x). \tag{23}$$

Category C

C1. Consider the family of linear multistep methods

$$u_{n+1} = \alpha u_n + \frac{h}{2}((2(1-\alpha)f_{n+1} + 3\alpha f_n - \alpha f_{n-1}) \tag{24}$$

where α is a real parameter.

1. Analyze consistency and order of the methods as functions of α, determining the value α^* for which the resulting method has maximal order.
2. Study the zero-stability of the method with $\alpha = \alpha^*$ by writing down its characteristic polynomial $\prod(r; \bar{h})$ where $\bar{h} = h\lambda$.
3. Propose a numerical method by which the region of absolute stability in the complex plane can be drawn.

C2. Define terms quoted in the following statements:

1. The "well-posedness" of the problem of solving $F(x; d) = 0$ where d is the parameter which the solution x depends on.
2. The "Butcher array" in an s-stage Runge-Kutta method.
3. The "Horner's method" for efficient evaluation of a polynomial at a given point.
4. A "least squares solution" for the linear equation $A\mathbf{x} = \mathbf{b}$.
5. The "shifted QR method" for eigenvalue computation.
6. The "singular value decomposition" of a rectangular matrix of size $m \times n$ where $m > n$.
7. The "Gershgorin circles" in locating eigenvalues of a square matrix.
8. The "Newton's formula" for polynomial interpolation.
9. That the semi-linear ordinary differential system $\dot{\mathbf{x}} = A\mathbf{x} + f(t, \mathbf{x})$ is a "stiff" system.
10. The "Quasi-Newton method" for solving the equation $f(x) = 0$.