Qualifying Examination in Numerical Analysis (580-780), January 2009

Direction: There are three categories of problems. Select one and only one problem from each category and answer all the subproblems. Use only one side of the paper to write your answers. Clearly identify the problem and all subproblem numbers on your answer sheets. The statements enclosed by (...) are meant to manifest the weight/purpose/application of the problem. You are not responsible for those statements.

Category A

A1. This question concerns solving a linear system of equations. Assume $A \in \mathbb{R}^{n\times n}$.

1. (10pts) Write down a pseudo-code in Matlab syntax that performs the LU decomposition of A *without pivoting* in BLAS2 (matrix-to-vector) operation. In your code, overwrite the upper triangular part of A by the U matrix and the strictly lower triangular part by the L matrix.
2. Assume that the first step of the Gaussian elimination process (without pivoting) has produced the following result,
$$A \Longrightarrow \begin{bmatrix} a_{11} & * \\ 0 & A_1 \end{bmatrix},$$
where $a_{11} \in \mathbb{R}$. Show that
   (a) (10pts) If A is symmetric and positive definite to begin with, then the reduced matrix A_1 must remain symmetric positive definite. (Therefore no pivoting is needed.)
   (b) (10pts) If A is strictly column diagonally dominant to begin with, i.e., for each k,
$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|$$
   then A_1 is also strictly column diagonally dominant. (Therefore no pivoting is needed.)
3. (10pts) Assuming $h > 0$ and f_{ij} are known scalars, write down a pseudo-code implementing the SOR(ω) method for the linear system of equations,
$$\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij}}{h^2} = f_{ij}, \quad i,j = 1,2,\cdots,n-1,$$
with zero boundary conditions, i.e., $U_{i,0} = U_{0,i} = U_{n,i} = U_{i,n} = 0$, for $i = 0,1,\cdots n$.

A2. This question concerns the condition number of a nonsingular matrix $A \in \mathbb{R}^{n\times n}$.

1. (10pts) Let B be a singular matrix. Show that
$$||A - B|| \geq ||A^{-1}||^{-1},$$
with respect to *any* given induced norm $\|\cdot\|$. (*Hint: Use the Banach lemma.*)
2. (10pts) Show that there exist two column vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$ and
$$\mathbf{y}^\top A^{-1}\mathbf{x} = ||A^{-1}||_2.$$
3. (10pts) Define
$$E := -||A^{-1}||_2^{-1}\mathbf{x}\mathbf{y}^\top.$$
Show that $A+E$ is singular and that $||E||_2 = ||A^{-1}||_2^{-1}$. (Therefore, the distance from the matrix A to the set of singular matrices in $\mathbb{R}^{n\times n}$, measured in the 2-norm, is precisely $||A||_2/\kappa_2(A)$.)

4. (10pts) (The assertions A2.2 and A2.3 are in fact true with respect to any induced norm.) Suppose now the procedure of Gaussian elimination with pivoting has been applied to A so that $PA = LU$, where P is a permutation matrix, L is unit lower triangular with elements $|l_{ij}| \le 1$ and U is upper triangular with elements u_{ij}. Show that

$$\kappa_\infty(A) \ge \|A\|_\infty / \min_j |u_{jj}|.$$

Category B

B1. Given a weight function $\omega(x) \ge 0$ over the interval $[-1, 1]$, define the inner product of two real-valued functions f and g by

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\omega(x)dx$$

and the norm $\|f\| := \langle f, f\rangle^{1/2}$, whenever these integrals exist. With respect to this inner product, an infinite sequence of orthogonal polynomials $\{p_k(x)\}$, $k = 0, 1, \ldots$, can be defined, i.e., $p_k(x)$ is a polynomial of degree precisely k and satisfies $\langle p_k, p_m\rangle = 0$ if $m \ne k$. For any function $f : [-1, 1] \to \mathbb{R}$ satisfying $\|f\| < \infty$, the series

$$\mathcal{F}(f) := \sum_{k=0}^{\infty} \widehat{f}_k p_k,$$

with coefficients

$$\widehat{f}_k := \frac{\langle f, p_k\rangle}{\langle p_k, p_k\rangle}, \tag{1}$$

is called the *generalized Fourier series* of f. For any positive integer n, define the polynomial

$$f_n(x) := \sum_{k=0}^{n} \widehat{f}_k p_k(x).$$

1. (10pts) Show *without* quoting the Parseval-Bessel theorem that $f_n(x)$ is the polynomial that minimizes $\|f - q\|$ among all polynomials $q(x)$ of degree less than or equal to n.
2. (10pts) Suppose that the integrals involved in $\widehat{f}_k$ are approximated by some quadrature rules. Let the resulting value be denoted by $\widetilde{f}_k$. Replacing the analytic Fourier coefficient $\widehat{f}_k$ by the discretized coefficient $\widetilde{f}_k$, how is this new polynomial

$$g_n(x) := \sum_{k=0}^{n} \widetilde{f}_k p_k(x)$$

related to the original f? (*Hint: Think about the analogy of DFT if the basis of polynomials are replaced by trigonometric functions.*)

3. (10pts) Show that if $p_{-1}(x) = 0$ and $p_1(x) = 1$, then the orthogonal polynomials satisfy a three-term recursive formula

$$p_{k+1}(x) = (x - \alpha_k)p_k(x) - \beta_k p_{k-1}(x), \quad k \ge 0.$$

Express α_k and β_k in terms of p_k and p_{k-1}.

4. (10pts) Show that all k roots of $p_k(x)$ are real, distinct, and in the interval $(-1, 1)$.

B2. This question concerns numerical ODE methods.

1. (10pts) Suppose the method

$$y_{n+2} + 9y_{n+1} - 10y_n = \frac{h}{2}(13f_{n+1} + 9f_n)$$

is applied to the initial value problem

$$y' = 0, \quad y(0) = c.$$

Let the starting values be $y_0 = c$ and $y_1 = c + \epsilon$, where ϵ is the machine precision. What values y_n are to be expected for a fixed step size h?

2. Let A be a constant diagonalizable matrix in $\mathbb{R}^{n\times n}$. Consider the initial value problem

$$\frac{d\mathbf{y}}{dt} = A\,\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{y}_0.$$

   (a) (10pts) Let Δt be a fixed step size and $\mathbf{y}^k$ denote an approximation to $\mathbf{y}(k\Delta t)$. Consider the θ-method applied to the above system

$$\begin{aligned} \mathbf{y}^{k+1} &= \mathbf{y}^k + \Delta t(\theta A\mathbf{y}^k + (1-\theta)A\mathbf{y}^{k+1}), \qquad k = 0, 1, \ldots, \\ \mathbf{y}^0 &= \mathbf{y}_0, \end{aligned}$$

   where $0 \le \theta \le 1$ and $\Delta t > 0$. Use the spectral information about A, i.e., the eigenvalues and eigenvectors, to give an explicit expression of $\mathbf{y}^k$.

   (b) (10pts) What is the domain of absolute stability for the above method? Be explicit.

   (c) (5pts) Are there values of θ for which the above method is A-stable? If yes, which ones? Justify.

   (d) (5pts) If $n = 2$ and $\lambda_1 = -1$ and $\lambda_2 = -10^5$ are the eigenvalues of A, what are the largest values of the time-step Δt that can be taken for $\theta = 0$ and for $\theta = 1$ respectively?

Category C

C1. This question concerns solving a nonlinear system of equations.

1. Given a sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$, consider the problem of finding a root for $f(x^*) = 0$.

   (a) (10pts) A method based on a *linear model* interpolates $f(x)$ by a straight line at two points a_k and b_k and then uses the zero of this model as the next iterate. The iterative scheme for this approach is

$$x_{k+1} = x_k - \frac{f(a_k)}{f[a_k, b_k]},$$

   where $f[a,b] := \frac{f(b)-f(a)}{b-a}$ denotes Newton's divide difference. Define the error $e_k := x^* - x_k$. Show that the error of an iterate from a linear model is

$$e_{k+1} = \frac{f[a_k, b_k, x^*]}{f[a_k, b_k]}(x^* - a_k)(x^* - b_k).$$

   (*Hint: Use Newton's divided difference interpolation formula.*)

   (b) (10pts) Suppose $f'(x^*) \neq 0$. Make an argument that the classical Newton method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

   is a special case of C1.1a, provided a_k and b_k are sufficiently close to x^*. Deduce from the error formula for the linear model that the Newton's iterative method has an error

$$e_{k+1} = -\frac{f''(\xi_1)}{2f'(\xi_2)}e_k^2,$$

   where ξ_1 and ξ_2 are points near x^* depending on k.

(c) (10pts) Suppose that f is a three times continuously differentiable function such that $f(x^*) = f'(x^*) = 0$ and $f''(x^*) \neq 0$. Show that if x_0 is sufficiently near to *but not equal to* x^*, then the iteration

$$x_{n+1} = x_n - 2f'(x_n)^{-1} f(x_n)$$

converges q-quadratically to x^*.

2. (10pts) Consider a system of nonlinear equations $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$. State formally the local convergence theorem for the Newton method.

C2. This question concerns Krylov subspace methods.

1. (10pts) What is the GMRES iteration? What quantity does it minimize?
2. (10pts) What is the CG iteration? What quantity does it minimize?
3. Suppose that A is a real-valued symmetric matrix with eigenvalues of -1, -100, and a cluster in the interval $[.9, 1.1]$.
   (a) (10pts) Suppose one solves $Ax = b$ with GMRES. How many iterations will it take to reduce the residual by a factor of 100?
   (b) (10pts) The CGNR iteration solves $Ax = b$ by applying CG to $A^T Ax = A^T b$. How many CGNR iterations will it take to reduce the residual by a factor of 100?