Conditioning

- A mathematical problem is said to be *well-conditioned* if small changes in the data of a problem result in small changes in the solution.
  - ⋄ Conditioning is a qualitative description of the sensitivity dependence of the solution to the change of data in a problem.
  - ⋄ An ill-conditioned problem can be solved accurately, if this possible at all, only by very careful calculation, quite aside from the method used. An unstable numerical method for a particular (even a well-conditioned) problem may give accurate results only in the early stage, but inevitably will give useless i results in the long run.
- An example: Root-finding of a polynomial could be an ill-conditioned problem.
  - ⋄ Consider using a machine with $\beta = 2$ and $t = 30$.
  - ⋄ The zeros of the 20-th degree polynomial

    $$p(x) = (x-1)(x-2)\ldots(x-20) = x^{20} - 210x^{19} + \ldots$$

    are obvious and well separated.
  - ⋄ Suppose a change in the 30-th significant base-2 digit is made only at the coefficient of x^{19}. Then, instead, the problem understood by the computer is to find roots of the perturbed polynomial,

    $$q(x) = p(x) + 2^{-23}x^{19}.$$

  - ⋄ Upon using very accurate method, the roots of $q(x)$ are

1.000000000	10.095266145 ± 0.643500904i
2.000000000	11.793633881 ± 1.652329728i
3.000000000	13.992358137 ± 2.518830070i
4.000000000	16.730737466 ± 2.812624894i
4.999999928	19.502429400 ± 1.940330347i
6.000000000	
6.999697234	
8.007267603	
8.917250249	
20.846908104.	

- ◇ Note that the small change in the coefficient -210 by a quantity $2^{-23} \approx 10^{-7}$ has caused ten of the zeros of $p(x)$ to become complex and that two have moved more than 2.81 units off the real axis. The problem is ill-conditioned.

• The sensitivity can sometimes be measured by the partial derivatives.

- ◇ Denote the dependence on α as

$$p(x, \alpha) = x^{20} - \alpha x^{19} + \ldots = 0.$$

- ◇ Use implicit differentiation. We have

$$\frac{\partial p(x, \alpha)}{\partial x} \frac{\partial x}{\partial \alpha} + \frac{\partial p(x, \alpha)}{\partial \alpha} = 0.$$

Thus,

$$\frac{\partial x}{\partial \alpha}|_{\alpha=210} = -\frac{\frac{\partial p}{\partial \alpha}}{\frac{\partial p}{\partial x}} = \frac{x^{19}}{\sum_{i=1}^{20} \prod_{j \neq i}(x-j)}.$$

- ◇ Evaluating this at each root gives $\frac{\partial x}{\partial \alpha}|_{x=i} = \frac{i^{19}}{\prod_{\substack{j=1 \\ j \neq i}}^{20}}(i-j)$. These numbers give a direct measure of the sensitivity of each of the roots to the coefficient α. It turns out, for example, $\frac{\partial x}{\partial \alpha}|_{x=18} \approx 1.0 \times 10^9$.