Polynomial Interpolation

- Two distinct points can uniquely determine a straight line. What can three points in a plane that are not collinear determine?
  - ◇ Given $\{(x_i, f_i)\}_{i=0}^{2}$, determine a quadratic polynomial

    $$p(t) - a_0 + a_1 t + a_2 t^2$$

    such that

    $$p(x_i) = f_i,\ i = 0, 1, 2.$$

  - ◇ The coefficients can be determined, in principle, by solving the linear equation

    $$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}.$$

  - ◇ Concerns in numerical calculation:
    - ▷ Is the system solvable?
    - ▷ How expensive?
    - ▷ How about conditioning of the linear system?
- The general interpolation problem:
  - ◇ Given points $\{(x_i, f_i)\}_{i=0}^{n}$, where x_i are distinct, determine a polynomial $p(t)$ satisfying

    $$\begin{aligned} \deg(p) &\le n, \\ p(x_i) &= f_i,\ i = 0, 1, \ldots, n. \end{aligned}$$

  - ◇ If $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$, then the interpolation problem is equivalent to solving the Vandermonde linear system

    $$\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}.$$