Lagrange Polynomials

- Can we construct n polynomials $\ell_j(t)$ for $j = 0, 1, \ldots, n$, each of which has degree n and does the following interpolation?

$$\ell_j(x_i) = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

  - ◇ The Lagrange polynomials defined by

$$\ell_j(t) := \prod_{i=0, i \neq j}^{n} \frac{t - x_i}{x_j - x_i}, \; j = 0, 1, \ldots, n$$

    do satisfy the desirable properties.

- For the general interpolation problem, the solution is given by

$$p(t) = \sum_{j=0}^{n} f_j \ell_j(t). \quad \text{(Why?)}$$

  - ◇ Note that we have explicitly answered the existence question raised in the interpolation theory.
  - ◇ The uniqueness of the interpolating polynomial follows from the (weak form of) Fundamental Theorem of Algebra, i.e., a polynomial of degree n vanishing at $n + 1$ distinct points is identically zero.
  - ◇ Note that we can only assure that $\deg(p) \leq n$, but not necessarily always $\deg(p) = n$. (Why?)

- The Lagrange polynomials provide useful insights into the approximation theory in general, but is difficult to apply in practice.
  - ◇ It is difficult to evaluate at an arbitrarily given point.
  - ◇ If is difficult to update when additional data are to be interpolated.