Polynomial Evaluation

- Given a polynomial in the *natural form*

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0,$$

  the evaluation of $\hat{p} = p(\hat{t})$ can be done stably by an algorithm called *synthetic division*:

```
p = a[n]
for i = n-1:-1:0
    p = p*t + a_[i]
end
```

  - ◇ Synthetic division requires only n additions and n multiplications. It is quite efficient.
  - ◇ Synthetic division is only quite *stable* in the sense that the computed value of $p(t)$ is the *exact* value of a polynomial $\tilde{p}$ whose coefficients differ from those of p by relative errors on the order of the rounding unit. (Students! Read this statement one more time. This is normally what is meant by *stability shown by backward error analysis.*
  - ◇ Note that the Lagrange polynomials are not in the natural form and hence is difficult to evaluate.

- We say a polynomial $p(t)$ is the *Netwon form* if

$$p(t) = c_0 + c_1(t-x_0) + c_2(t-x_0)(t-x_1) + \ldots + c_n(t-x_0)(t-x_1)\ldots(t-x_{n-1}).$$

  - ◇ Evaluation of a Newton form is easy :

```
p = c[n]
for i = n-1:-1:0
    p = p*(t-x[i]) + c[i]
end
```

  - ◇ It remains to determine the coefficients $c_0, \ldots, c_n$ so that $p(t)$ interpolates the data $\{(x_i, f_i)\}$ for $i = 0, 1, \ldots, n$.

Determining the Newton Form

- The coefficients of the Newton form of an interpolant can be determined through the system

$$
\begin{aligned}
f_0 &= c_0 \\
f_1 &= c_0 + c_1(x_1 - x_0) \\
&\vdots \\
f_n &= c_0 + c_1(x_n - x_0) + \ldots + c_n(x_n - x_0)\ldots(x_n - x_{n-1}).
\end{aligned}
$$

  ◇ This is a lower triangular system whose diagonal elements are nonzero, if all given nodes are distinct.

$$
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & (x_1 - x_0) & 0 & \ldots & 0 \\
1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) & \ldots & 0 \\
\vdots & \vdots & & & \vdots \\
1 & (x_n - x_0) & (x_n - x_0)(x_n - x_1) & \ldots & (x_n - x_0)\ldots(x_n - x_{n-1})
\end{bmatrix}
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}
=
\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}
$$

  ◇ If new points are added to be interpolated, the coefficients already determined will *not* be affected. We just need to add a new row to determine c_{n+1}.

- There is a yet better way, called the Newton divided differences, to determine the coefficients.