Newton's Interpolation Formula

- Newton's interpolation formula is mathematically equivalent to the Lagrange's formula, but is much more efficient.
  - ◇ One of the most important features of Newton's formula is that one can gradually increase the support data without recomputing what is already computed.
- Divided Difference:
  - ◇ Let $P_{i_0 i_1 \ldots i_k}(t)$ represent the k-th degree polynomial that satisfies

$$P_{i_0 i_1 \ldots i_k}(x_{i_j}) = f_{i_j} \tag{1}$$

    for all $j = 0, \ldots, k$.
  - ◇ The recursion formula holds:

$$p_{i_0 i_1 \ldots i_k}(t) = \frac{(t - x_{i_0}) P_{i_1 \ldots i_k}(t) - (t - x_{i_k}) P_{i_0 \ldots i_{k-1}}(t)}{x_{i_k} - x_{i_0}} \tag{2}$$

    - ▷ The right-hand side of (**??**), denoted by $R(t)$, is a polynomial of degree $\leq k$.
    - ▷ $R(x_{i_j}) = f_{i_j}$ for all $j = 0, \ldots, k$. That is, $R(t)$ interpolates the same set of data as does the polynomial $P_{i_0 i_1 \ldots i_k}(t)$.
    - ▷ By uniqueness, $R(t) = P_{i_0 i_1 \ldots i_k}(t)$.
  - ◇ The difference $P_{i_0 i_1 \ldots i_k}(t) - P_{i_0 i_1 \ldots i_{k-1}}(t)$ is a k-th degree polynomial that vanishes at x_{i_j} for $j = 0, \ldots, k-1$. Thus we may write

$$\begin{aligned} P_{i_0 i_1 \ldots i_k}(t) &= P_{i_0 i_1 \ldots i_{k-1}}(t) \\ &+ f_{i_0 \ldots i_k}(t - x_{i_0})(t - x_{i_1}) \ldots (t - x_{i_{k-1}}). \end{aligned} \tag{3}$$

◇ The leading coefficients $f_{i_0 \dots i_k}$ can be determined recursively from the formula (**??**), i.e.,

$$f_{i_0 \dots i_k} = \frac{f_{i_1 \dots i_k} - f_{i_0 \dots i_{k-1}}}{x_{i_k} - x_{i_0}} \tag{4}$$

where $f_{i_1 \dots i_k}$ and $f_{i_0 \dots i_{k-1}}$ are the leading coefficients of the polynomials $P_{i_1 \dots i_k}(x)$ and $P_{i_0 \dots i_{k-1}}(x)$, respectively.

- Let $x_0, \dots, x_k$ be support arguments (but not necessarily in any order) over the interval $[a, b]$. We define the Newton's divided difference as follows:

$$f[x_0] : = f(x_0) \tag{5}$$

$$f[x_0, x_1] : = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \tag{6}$$

$$f[x_0, \dots, x_k] : = \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0} \tag{7}$$

- The k-th degree polynomial that interpolates the set of support data $\{(x_i, f_i) | i = 0, \dots, k\}$ is given by

$$\begin{aligned} P_{x_0 \dots x_k}(x) &= f[x_0] + f[x_0, x_1](x - x_0) \\ &+ \dots + f[x_0, \dots, x_k](x - x_0)(x - x_1) \dots (x - x_{k-1}). \end{aligned} \tag{8}$$