Error in Interpolation

- Suppose the polynomial $p(t)$ interpolates a function $f(t)$ at nodes $t = x_0, x_1, \ldots, x_n$, i.e., suppose $p(x_i) = f_i = f(x_i)$ for all $i = 0, 1, \ldots, n$. Define
$$e(t) = f(t) - p(t)$$
as the error function. What can be said about the behavior of $e(t)$?
  - ◇ Choose $x_{-1} \neq x_i$ for any $i = 0, 1, \ldots, n$. Define
$$F(u) = f(u) - p(u) - (f(x_{-1}) - p(x_{-1}))\frac{\prod_{i=0}^{n}(x - x_i)}{\prod_{i=0}^{n}(x_{-1} - x_i)}.$$
  - ◇ Observe $F(x_i) = 0$ for $i = -1, 0, 1, \ldots n$, i.e., $F(u)$ has $n+2$ zeros.
  - ◇ By Rolle's theorem, there exists ξ betweein $x_{-1}, x_0, \ldots, x_n$ such that $F^{(n+1)}(\xi) = 0$.
  - ◇ Note that
$$f^{(n+1)}(\xi) - (f(x_{-1}) - p(x_{-1}))\frac{(n+1)!}{\prod_{i=0}^{n}(x_{-1} - x_i)}.$$
  - ◇ Since x_{-1} is arbitrary, we have established
$$e(t) = \frac{\prod_{i=0}^{n}(t - x_i)}{(n+1)!} f^{(n+1)}(\xi) \qquad (1)$$
for some ξ between $t, x_0, \ldots, x_n$.
    - ▷ Note that $\xi = \xi(t)$ varies as t varies.