Hermite Interpolation

- Thus far, the interpolation has been required only to interpolate the functional values. Sometimes it is desirable that the derivatives are also interpolated.
- Given $\{x_i\}, i = 1, \ldots k$ and values $a_i^{(0)}, \ldots, a_i^{(r_i)}$ where r_i are nonnegative integers. We want to construct a polynomial $p(t)$ such that

  $$P^{(j)}(x_i) = a_i^{(j)} \tag{1}$$

  for $i = 1, \ldots, k$ and $j = 0, \ldots, r_i$.

  - ◇ Such a polynomial is called an *osculatory (kissing) interpolating polynomial* of a function f if $a_i^{(j)} = f^{(j)}(x_i)$ for all i and j.
- Some examples of osculatory interpolation:
  - ◇ Suppose $r_i = 0$ for all i. Then this is simply the ordinary Lagrange or Newton interpolation.
  - ◇ Suppose $k = 1, x_1 = a, r_1 = n - 1$, then the osculatory polynomial becomes $p)t) = \sum_{j=0}^{n-1} f^{(j)}(a)\frac{(t-a)^j}{j!}$ which is the Taylor's polynomial of f at $x = a$.
- One of the most interesting osculatory interpolations is when $r_i = 1$ for all $i = 1, \ldots, k$. That is, the values of $f(x_i)$ and $f'(x_i)$ are to be interpolated.
  - ◇ The resulting $(2k - 1)$-degree polynomial is called the *Hermite interpolating polynomial.*
  - ◇ Very useful for deriving numerical integration scheme of high precision.

Construction of Hermite Polynomial

- Recall the $(k-1)$-degree polynomial

$$\ell_i(t) = \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{t - x_j}{x_i - x_j}$$

has the property

$$\ell_i(x_j) = \delta_{ij}.$$

- Define

$$\begin{aligned} h_i(t) &= [1 - 2(t - x_i)\ell_i'(x_i)]\ell_i^2(t) \quad (2) \\ g_i(t) &= (t - x_i)\ell_i^2(x). \quad (3) \end{aligned}$$

  - ◇ Note that both $h_i(x)$ and $g_i(x)$ are of degree $2k - 1$.
  - ◇ The following properties can be checked out:

$$\begin{aligned} h_i(x_j) &= \delta_{ij}; \\ g_i(x_j) &= 0; \\ h_i'(x_j) &= [1 - 2(t - x_i)\ell_i'(x_i)]2\ell_i(t)\ell_i'(t) - 2\ell_i'(x_i)\ell_i^2(t)|_{t=x_j} = 0; \\ g_i'(x_j) &= (x - x_i)2\ell_i(x)\ell_i'(x) + \ell_i^2(x)|_{x=x_j} = \delta_{ij}. \end{aligned}$$

- The Hermite polynomial can be written down as

$$p(t) = \sum_{i=1}^{k} f(x_i)h_i(t) + f'(x_i)g_i(t)). \quad (4)$$