Newton-Cotes Formula

- The way the trapezoidal rule is derived can be generalized to higher degree polynomial interpolants. Such a quadrature rule is called a *Newton-Cotes formula.*

- Let $x_0, x_1, \ldots, x_n$ be given nodes in $[a, b]$. Recall that the Lagrange interpolation of a function at these nodes is given by the polynomial

$$p(t) = \sum_{j=0}^{n} f(x_j)\ell_j(t)$$

  where each $\ell_j(t)$ is the Lagrange polynomial

$$\ell_j(t) := \prod_{i=0, i\neq j}^{n} \frac{t - x_i}{x_j - x_i}, \; j = 0, 1, \ldots, n.$$

- We therefore have

$$\int_a^b f(t)dt \approx \int_a^b p(t)dt = \sum_{j=0}^{n} \omega_j f(x_j)$$

  where the weight ω_j is determined by

$$\omega_j = \int_a^b \ell_j(t)dt. \tag{1}$$

  ◇ Note that if $f(t)$ itself is a polynomial of degree $\leq n$, then

$$f(t) = \sum_{j=0}^{n} f(x_j)\ell_j(t). \text{ (Why?)}$$

  In this case, the Newtow-Cotes quadrature rule evaluates $\int_a^b f(t)dt$ precisely.

  ◇ The Newton-Cotes quadrature rule has degree of precision at least n.

An Example — Simpson's Rule

- It is nice to know how the weight ω_j should be calculated. However, there are some other concerns:
  - ◇ These weights are difficult to evaluate.
  - ◇ How high can the degree of precision be pushed?
- Suppose we approximate $f(t)$ by a quadratic polynomial $p_2(t)$ that interpolates $f(a), f\left(\frac{a+b}{2}\right)$ and $f(b)$.
  - ◇ It can be shown that (I derived it in class)

$$Q_3(f) = \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]. \tag{2}$$

  - ◇ The error part is tricky!
    - ▷ Observe that the *function (in variable s)* from Newton's formula, $g(s) = p_2(s) + f[a, b, \frac{a+b}{2}, t](s-a)(s-b)(s-\frac{a+b}{2})$, interpolates f at $a, b, \frac{a+b}{2}$ and t.
    - ▷ Thinking t as arbitrary, we should have

$$f(t) = p_2(t) + f[a, b, \frac{a+b}{2}, t](t-a)(t-b)(t-\frac{a+b}{2})$$

    - ▷ The error $E_3(f)$ therefore is given by

$$E_3(f) = \int_a^b f[a, b, \frac{a+b}{2}, t]\omega(t)dt \tag{3}$$

      with $\omega(t) := (t-a)(t-b)(t-\frac{a+b}{2})$.
    - ▷ We already know that the degree of precision is ≥ 2. Can this be better?

- The function $\omega(x)$ changes sign as x crosses $\frac{a+b}{2}$. So we have to analyze $E_3(x)$ by a different approach.
  - ◇ Let $\Omega(x) := \int_a^x \omega(t)dt$. Then $\Omega'(x) = \omega(x)$.
  - ◇ By integration by parts, we have

$$E_3(f) = f[a, b, \frac{a+b}{2}, x]\Omega(x)|_a^b - |_a^b f[a, b, \frac{a+b}{2}, x, x]\Omega(x)dx.$$

  - ◇ Observe that $\Omega(a) = \Omega(b) = 0$. Observe also that $\Omega(x) > 0$ for all $x \in (a, b)$.
  - ◇ We may apply the mean value theorem to conclude that

$$\begin{aligned} E_3(f) &= -\int_a^b f[a, b, \frac{a+b}{2}, x, x]\Omega(x)dx \\ &= -f[a, b, \frac{a+b}{2}, \xi, \xi] \int_a^b \Omega(x)dx \\ &= -\frac{f^{(4)}(\eta)}{4!}\frac{4}{15}\left(\frac{b-a}{2}\right)^5 \\ &= -\frac{f^{(4)}(\eta)}{90}\left(\frac{b-a}{2}\right)^5. \end{aligned}$$

  - ◇ The degree of precision for Simpson's rule is 3 rather than 2.

- Divide the interval into $2n$ equally space subintervals with $h = \frac{b-a}{2n}$ and $x_i = a + ih$ for $i = 0, 1, \ldots, 2n$. Upon applying Simpson's rule over *two* consecutive subintervals $[x_{2j}, x_{2j+2}]$ for $j = 0, 1, \ldots, n-1$ and summing up these integrals, we obtain the composite Simpson's rule:

$$\int_a^b f(t)dt \approx \frac{h}{3}\{f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}\}. \tag{4}$$

  - ◇ The error formula for the composite Simpson's rule can be obtained in the same way as we derived the error formula for the composite trapezoidal rule:

$$E_{3,cs} = -\frac{(b-a)h^4}{180}f^{(4)}(\eta). \tag{5}$$