How to Handle Singularities

- An improper integral such as $\int_0^1 \frac{1}{\sqrt{t}} dt$ does exist even though it is not defined at $t = 0$.
  - ◇ A Newton-Cotes quadrature must exclude the node $t = 0$. Even so, the results are not good. (Why?)
  - ◇ A better approach is to incorporate the singularity into the quadrature rule itself.
- The idea is to integrate a function $f(t)$ with respect to a specified weight function $w(x)$, i.e.,

$$I_w(f) := \int_a^b f(t)w(t)dt \tag{1}$$

  where $w(t) \geq 0$ on $[a, b]$. In such a case, we consider a quadrature rule of the form

$$Q_w(f) := \sum_{i=1}^{n} \alpha_i f(x_i). \tag{2}$$

  - ◇ Note that the weight function $w(x)$ does not appear on the right hand side of (**??**).
- An example: Suppose $w(t) = t^{-1/2}$ and suppose the nodes selected are $x_1 = \frac{1}{4}$ and $x_2 = \frac{3}{4}$. We want to determined the coefficients α_1 and α_2 in the quadrature so that

$$\begin{aligned} \int_0^1 1t^{-1/2}dt = 2 &= \alpha_1 + \alpha_2 \\ \int_0^1 tt^{-1/2}dt = \frac{2}{3} &= \frac{1}{4}\alpha_1 + \frac{3}{4}\alpha_2. \end{aligned}$$

  - ◇ The newly weighted quadrature reads like

$$\int_0^1 f(t)t^{-1/2}dt \approx \frac{5}{3}f(\frac{1}{4}) + \frac{2}{3}f(\frac{3}{4}).$$

- ◇ As a comparison, the unweighted Newton-Cotes formula, which is the trapezoidal rule, reads like

$$\int_0^1 f(t)t^{-1/2}dt \approx \frac{1}{2}f(\frac{1}{4}) + \frac{1}{2}f(\frac{3}{4}).$$

- Try to approximate the integral

$$\int_0^x \left(\frac{\cos t}{2\sqrt{t}} - \sqrt{t}\sin t\right) dt = \cos x\sqrt{x}$$

by the quadrature rules you know about and compare the results.

- The entire theory that works for Newton-Cotes or Gaussian quadratures can be generalized to the weighted integrals. The arguments will not be repeated in this course, but take note of this possibility.