Orthogonal Polynomials

- Two functions $f$ and $g$ defined on $[a, b]$ are said to be orthogonal if and only if
  \[ \langle f, g \rangle := \int_{a}^{b} f(t)g(t)dt = 0. \]  
  (1)

- The operation $\langle f, g \rangle$ may be regarded as an inner product of $f$ and $g$.

- A sequence \( \{p_i(t)\}_{i=0}^{\infty} \) of polynomials with $\deg(p_i) = i$ is called a sequence of orthogonal polynomials on $[a, b]$ if
  \[ \int_{a}^{b} p_i(t)p_j(t)dt = 0, \text{ whenever } i \neq j. \]

- The orthogonality is not affected by scalar multiplication. We may assume that all $p_i(t)$ are monic, i.e., the leading coefficients of all $p_i(t)$ are one.

- Any $n$-th degree polynomial $q(t)$ can uniquely be written as
  \[ q(t) = b_n p_n(t) + b_{n-1} p_{n-1}(t) + \ldots + b_0 p_0(t). \]

  That is, the orthogonal polynomials of degree $\leq n$ span the entire space of polynomials of degree $\leq n$. 
Constructing Orthonomial Polynomials

- The following process constructs orthogonal polynomials over an arbitrary \([a, b]\):
  - \(p_0(t) = 1\).
  - \(p_1(t) = t - a_0\), but
    \[
    0 = \langle p_0, p_1 \rangle \implies a_0 = \frac{b + a}{2}.
    \]
  - Suppose \(p_0(t), \ldots, p_n(t)\) has been constructed. We seek \(p_{n+1}(t)\) in the form
    \[
    p_{n+1}(t) = (t - a_{n+1})p_n + b_{n+1}p_{n-1}(t) + c_{n+1}p_{n-2}(t) + \ldots
    \]
    △ \(a_{n+1}\) can be determined from
    \[
    0 = \langle p_{n+1}, p_n \rangle \implies a_{n+1} = \frac{\langle t, p_n^2 \rangle}{\langle p_n, p_n \rangle}.
    \]
    △ \(b_{n+1}\) can be determined from
    \[
    0 = \langle p_{n+1}, p_{n-1} \rangle \implies b_{n+1} = \frac{\langle t, p_n p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle}.
    \]
    △ \(c_{n+1}\) can be determined from
    \[
    0 = \langle p_{n+1}, p_{n-2} \rangle \implies c_{n+1} = \frac{\langle t, p_n p_{n-2} \rangle}{\langle p_{n-2}, p_{n-2} \rangle}.
    \]
    But, surprisingly, the denominator \(\langle t, p_n p_{n-2} \rangle = \langle t p_{n-1}, p_n \rangle = 0\). (Why?)

- We therefore conclude that \(p_{n+1}(t)\) can be generated by a \textit{three-term recurrence} formula:
  \[
  p_{n+1}(t) = (t - a_{n+1})p_n(t) + b_{n+1}p_{n-1}(t).
  \]
  (2)

- If \([a, b] = [-1, 1]\), such a sequence of orthogonal polynomials are called the Legendre polynomials.