Orthogonal Polynomials

- Two functions f and g defined on $[a, b]$ are said to be *orthogonal* if and only if
$$\langle f, g\rangle := \int_a^b f(t)g(t)dt = 0. \tag{1}$$
  - ◇ The operation $\langle f, g\rangle$ may be regarded as an *inner product* of f and g.
  - ◇ A sequence $\{p_i(t)\}_{i=0}^{\infty}$ of polynomials with $\deg(p_i) = i$ is called a *sequence of orthogonal polynomials on* $[a, b]$ if
$$\int_a^b p_i(t)p_j(t)dt = 0, \quad \text{whenever } i \neq j.$$
    - ▷ The orthogonality is not affected by scalar multiplication. We may assume that all $p_i(t)$ are monic, i.e., the leading coefficients of all $p_i(t)$ are one.
    - ▷ Any n-th degree polynomial $q(t)$ can uniquely be written as
$$q(t) = b_n p_n(t) + b_{n-1}p_{n-1}(t) + \ldots + b_0 p_0(t).$$
That is, the orthogonal polynomials of degree $\leq n$ span the entire space of polynomials of degree $\leq n$.

Constructing Orthonomial Polynomials

- The following process constructs orthogonal polynomials over an arbitrary $[a, b]$:
  - ◇ $p_0(t) = 1$.
  - ◇ $p_1(t) = t - a_0$, but
    $$0 = \langle p_0, p_1 \rangle \Longrightarrow a_0 = \frac{b+a}{2}.$$
  - ◇ Suppose $p_0(t), \ldots p_n(t)$ has been constructed. We seek $p_{n+1}(t)$ in the form
    $$p_{n+1}(t) = (t - a_{n+1})p_n + b_{n+1}p_{n-1}(t) + c_{n+1}p_{n-2}(t) + \ldots.$$
    - ▷ a_{n+1} can be determined from
      $$0 = \langle p_{n+1}, p_n \rangle \Longrightarrow a_{n+1} = \frac{\langle t, p_n^2 \rangle}{\langle p_n, p_n \rangle}.$$
    - ▷ b_{n+1} can be determined from
      $$0 = \langle p_{n+1}, p_{n-1} \rangle \Longrightarrow b_{n+1} = \frac{\langle t, p_n p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle}.$$
    - ▷ c_{n+1} can be determined from
      $$0 = \langle p_{n+1}, p_{n-2} \rangle \Longrightarrow c_{n+1} = \frac{\langle t, p_n p_{n-2} \rangle}{\langle p_{n-2}, p_{n-2} \rangle}.$$

      But, surprisingly, the denominator $\langle t, p_n p_{n-2} \rangle = \langle t p_{n-1}, p_n \rangle = 0$. (Why?)
- We therefore conclude that $p_{n+1}(t)$ can be generated by a *three-term recurrence* formula:
  $$p_{n+1}(t) = (t - a_{n+1})p_n(t) + b_{n+1}p_{n-1}(t). \qquad (2)$$
- If $[a, b] = [-1, 1]$, such a sequence of orthogonal polynomials are called the Legendre polynomials.