Deriving the Gaussian Quadrature

- We are interested in deriving quadratures with higher degrees of precision. Toward this end, observe the following three facts:
  - ◇ *If A quadrature formula using n distinct abscissas degree of precision $\geq n-1$, then it must result from the integration of an interpolation polynomial.*
    - ▷ Suppose the quadrature rule is $Q_n(f) = \sum_{i=1}^{n} \alpha_i f(x_i)$.
    - ▷ Then $\sum_{i=1}^{n} \alpha_i x_i^k = \frac{b^{k+1} - a^{k+1}}{k+1}$ for $k = 0, \ldots, n-1$.
    - ▷ We may rewrite this system of equations as

$$\begin{bmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \\ \vdots & & \\ x_1^{n-1} & \ldots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b-a \\ \frac{b^2-a^2}{2} \\ \vdots \\ \frac{b^n-a^n}{n} \end{bmatrix}. \tag{1}$$

      The coefficient matrix is the Vandermonde matrix. Thus THE system (**??**) has a unique solution for α_i), $i = 1, \ldots, n$.
    - ▷ On the other hand, we can construct an interpolating polynomial of degree $n-1$ using the same nodes $\{x_i\}_{i=1}^n$.
    - ▷ This polynomial results in a quadrature formula $\sum_{i=1}^{n} \beta_i f(x_i)$ which has degree of precision at least $n-1$.
    - ▷ By setting $E_n(x^k) = 0$ for $k = 0, \ldots, n-1$ in the new quadrature, we end up with the same linear system (**??**).
    - ▷ By uniqueness, it must be that $\alpha_i = \beta_i$.

- ◇ Since the Gaussian quadrature is expected to have degree of precision higher than n, we conclude that
  - ▷ It may be thought of as the integration of a certain polynomial that interpolates $f(x)$ at a certain set of nodes $x_1, \ldots, x_n$.
  - ▷ It must remain true that

$$E_n(f) = \int_a^b f[x_1, \ldots, x_n, t] \prod_{i=1}^{n} (t - x_i) dt. \qquad (2)$$

  - ▷ Not only $E_n(x^k) = 0$ for $k = 0, \ldots, n-1$, but we further would like to require $E_n(x^k) = 0$ for $k = n, \ldots, n + \nu$, and for ν as large as possible.
- ◇ *If $f(t) = t^{n+\nu}, \nu \geq 0$, then the n-th divided difference $f[x_1, \ldots, x_n, t]$ is a polynomial of degree at most ν.*
  - ▷ When $n = 1$, we find $f[x_1, t] = \frac{f(x_1) - f(t)}{x_1 - t} = \frac{x_1^{1+\nu} - t^{1+\nu}}{x_1 - t}$ is obviously a polynomial of degree ν.
  - ▷ Suppose the assertion is true for $n = k$. Consider $f(t) = t^{k+1+\nu}$. (We are preparing to use the Induction Principle on n.)
  - ▷ Regard $f(t) = t^{k+(1+\nu)}$. Then, by induction hypothesis, the difference quotient $f[x_2, \ldots, x_{k+1}, t]$ is a polynomial of degree at most $\nu + 1$.
  - ▷ Observe that

$$f[x_1, \ldots, x_{k+1}, t] = \frac{f[x_1, \ldots, x_{k+1}] - f[x_2, \ldots, x_{k+1}, t]}{x_1 - t}.$$

    Note that the numerator has a zero at $t = x_1$. After cancelation, $f[x_1, \ldots, x_{k+1}, t]$ is a polynomial of degree of most ν.
  - ▷ The assertion now follows from the induction.

- With all said, if we choose the nodes x_i so that $\omega(t) = \prod_{i+1}^{n} (t - x_i)$ is perpendicular to all lower degree polynomials, then the error $E(x^{n+\nu})$ would be zero for $\nu = 0, 1, \ldots n - 1$. The theory of orthogonal polynomials now kicks in.