Finally, The Gaussian Quadrature!

- Recall that
  - ◇ The Gaussian quadrature is a special type of Newton-Cotes quadrature.
  - ◇ The error in the quadrature is given by

$$E_n(f) = \int_a^b f[x_1, \ldots, x_n, t] \prod_{i=1}^{n} (t - x_i) dt.$$

  - ◇ The divided difference $f[x_1, \ldots, x_n, t]$ is a polynomial of degree ν is $f(t) = t^{n+\nu}$.
  - ◇ If we can choose the nodes x_i so that $\omega(t) = \prod_{i+1}^{n} (t - x_i)$ is the n-th orthogonal polynomials, then the error $E(t^{n+\nu} = 0$ for all $\nu = 0, 1, \ldots, n-1$¿
- Need to make sure all roots of the n-th orthogonal polynomial are real-valued, distinct, and contained in (a, b).
  - ◇ Let $x_1, \ldots, x_m \in (a, b)$ denote all the distinct, real zeros of $\phi_n(x)$ with odd multiplicity.
  - ◇ Assume $m < n$. We want to prove by contradiction.
  - ◇ Consider the integral $\int_a^b (x - x_1) \ldots (x - x_m) \phi_n(x) dx$. Note that the integrand does not change sign over $[a, b]$ and is not identically zero. Thus the integral is positive.
  - ◇ But $(x - x_1) \ldots (x - x_m)$ is a polynomial of degree $m < n$.
  - ◇ By orthogonality of $\omega_n(x)$, the integral should be zero. This is a contradiction.
  - ◇ It must be that $m = n$, and the multiplicity is 1.

- Gaussian abscissas and weights over the interval $[-1, 1]$.

n	Abscissas x_j	Weights α_j
2	$\pm 1/\sqrt{3}$	1
3	$\pm\sqrt{0.6}$	5/9
	0	8/9
4	± 0.8611363116	0.3478548451
	± 0.3399810436	0.6521451549
5	± 0.9061798459	0.2369268850
	± 0.5384693101	0.4786286705
	0	0.5688888889
6	± 0.9324695142	0.1713244924
	± 0.6612093865	0.3607615730
	± 0.2386191861	0.4679139346

- To integrate $f(x)$ over an arbitrary interval $[a, b]$,
  - ◇ Change of variables:

$$x = a + \frac{b-a}{2}(\xi + 1), \; dx = \frac{b-a}{2} d\xi.$$

  - ◇ Substitution:

$$\begin{aligned} \int_a^b f(x)dx &= \int_{-1}^{1} f(a + \frac{b-a}{2}(\xi+1))\frac{b-a}{2} d\xi \\ &= \frac{b-a}{2}\int_{-1}^{1} f(a + \frac{b-a}{2}(\xi+1))d\xi \\ &\approx \frac{b-a}{2}\sum_{i=1}^{n} \alpha_i f(x_i) \end{aligned}$$

    - ▷ α_j are the tabulated Gaussian weights associated with the tabulated Gaussian abscissa a_j in $[-1, 1]$.
    - ▷ x_j is obtained from a_j through $x_j = a + \frac{b-a}{2}(a_j + 1), \; j = 1, \ldots, n$.