Using Numerical Differentiation

- Consider the two-point boundary value problem

$$-y^{''} = f(x,y) \quad (1)$$
$$y(a) = \alpha \quad (2)$$
$$y(b) = \beta. \quad (3)$$

  - ◇ Let the interval $[a,b]$ be partitioned into $a = x_0 < x_1 < \ldots < x_N = b$ where $x_i = a + ih$, $i = 0, 1, \ldots, N$ and $h = \frac{b-a}{N}$.
  - ◇ Let

$$y(x_i) \approx y_i,$$
$$y^{''}(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}.$$

- The differential equation (1) is discretized and the solution is approximated by solving the algebraic equation of the form

$$Jy + h^2 F(y) = b. \quad (4)$$

  - ◇ The solution $y(x)$ is understood as the column vector $[y_1, \ldots, y_{N-1}]^T$.
  - ◇ The matrix J is $(N-1) \times (N-1)$ and tri-diagonal,

$$J = \begin{bmatrix} -2 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & -2 & 1 \\ 0 & & & & 1 & -2 \end{bmatrix}$$

  - ◇ The nonlinear function $F(y)$ is a $(N-1) \times 1$ column vector,

$$F(y) = [f(x_1, y_1), \ldots, f(x_{N-1}, y_{N-1})]^T.$$

  - ◇ The constant vector contains the boundary information,

$$b = [-\alpha, 0, \ldots, 0, -\beta]^T.$$

- The algebraic equation can now be solved by Newton's iterative method.