Solving Ordinary Differential Equations

- Differential equation in the normal form:

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

  where $x \in R$ is the independent variable, $y \in R^n$ is the dependent variable, and $f : R \times R^n \longrightarrow R^n$.

- A general m-th order ODE

$$\frac{d^m y}{dx^m} = f(x, y, \frac{dy}{dx}, \ldots, \frac{d^{m-1}y}{dx^{m-1}})$$

  with $x, y \in R$ can be reduced to the system (**??**).

  - ◇ Define

$$\begin{aligned} y_1 &:= y \\ y_2 &:= \frac{dy_1}{dx} = \frac{dy}{dx} \\ y_3 &:= \frac{dy_2}{dx} = \frac{d^2y}{dx^2} \\ &\vdots \\ y_m &:= \frac{dy_{m-1}}{dx} = \frac{d^{m-1}y}{dx^{m-1}} \end{aligned}$$

  - ◇ The new system is given by

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ \vdots \\ y_{m-1} \\ y_m \end{bmatrix} = \begin{bmatrix} y_2 \\ \vdots \\ y_m \\ f(x, y_1, \ldots, y_{m-1}) \end{bmatrix}$$

An Example

- Consider the differential system

$$\begin{aligned} \frac{dy_1}{dt} &= ay_1 + by_1y_2 \\ \frac{dy_2}{dt} &= cy_2 + dy_1y_2, \end{aligned}$$

  each $y_i(t)$ representing the population of a certain spices subject to the interaction with the other species.

  - ◇ The case, $a > 0$, $c < 0$, $b < 0$, and $d > 0$, represents that y_1 is a prey while y_2 is a predator (in a broad sense.)
  - ◇ The case, $a > 0$, $c > 0$, $b < 0$, and $d < 0$, represents that both y_1 and y_2 can survive without the other, and that they are competing.
  - ◇ The case, $a > 0$, $c > 0$, $b > 0$, and $d > 0$, represents that both y_1 and y_2 can survive without the other, and that their co-existence helps each other.
  - ◇ The case, $a < 0$, $c < 0$, $b > 0$, and $d > 0$, represents that without the other species, both y_1 and y_2 will extinct, and that they are mutually dependent.

- The suitability of a model needs to be tested against observation from real phenomenon before it can be accepted as a general model.

  - ◇ It is possible that the model is overly simplified. Need to modify the model, or to develop a more complicated model.
  - ◇ It is possible that the values of the parameters are not valid. Need to re-estimate the parameters.
  - ◇ The process of designing of a model, comparing the solution from the model with real data, and re-designing usually needs to be iterated several times.