Euler's Method

- We are interested in knowing not just the qualitative behavior of the solution, but also solving the differential equation quantitatively. Numerical methods are the necessary tools.

- The simplest numerical method,

  ◇ If $y(x)$ is a solution, then

  $$y(x_n + h) = y(x_n) + hy^{'}(x_n) + O(h^2)$$

  is the Taylor series expansion of $y(x_n + h)$ near x_n.

  ◇ Suppose the *accepted* solution at x_n is given $y(x_n) \approx y_n$, then the truncated Taylor series suggests that

  $$y(x_n + h) \approx y_{n+1} = y_n + hf(x_n, y_n) \tag{1}$$

  should be a reasonable approximation.

- Questions always asked in numerical ODE:

  ◇ What is the magnitude of the *global error*

  $$e_n := y_n - y(x_n) \tag{2}$$

  at the n-th step? How does the error propagate?

  ◇ How does the step size h affect the accuracy?

  ◇ What kinds of errors are involved in the calculation? How do they affect the overall accuracy? How to control the error to get the best possible accuracy?

Error Analysis

- The *local truncation error* is defined to be the difference between the exact solution $y(x_{n+1})$ and the approximate solution y_{n+1}, provided no previous errors have been introduced into the numerical scheme.
  - ◇ For Euler's method, the LTE is
  $$T_{n+1} := y(x_{n+1}) - y(x_n) - hf(x_n, y(x_n)). \tag{3}$$
  - ◇ Suppose $y''(x)$ is continuous and is bounded by C Then
  $$T_{n+1} = \frac{h^2}{2}|y''(\xi)| \leq \frac{C}{2}h^2.$$
- The (global) errors produced at the previous step will be passed on to the next step. To see how the errors are propagated, subtract (**??**) from (**??**). The global error at the next step is given by
  $$e_{n+1} = e_n + h[f(x_n, y_n) - f(x_n, y(x_n))] - T_{n+1} \tag{4}$$
  where we have assumed that f satisfies the Lipschitz condition in y with constant L.
  - ◇ It follows that
  $$|e_{n+1}| \leq (1 + hL)|e_n| + T \tag{5}$$
  where $T = \max |T_n| = O(h^2)$.
  - ◇ Note that the growth factor $1 + hL$ determines how e_n gets propagated.
  - ◇ The formula can be applied repeatedly to give
  $$|e_n| \leq T\frac{(1 + hL)^n - 1}{hL} + (1 + hL)^n|e_0| \tag{6}$$
- Look at the example $y' = \lambda y$.
  $$\begin{aligned} e_{n+1} &= (1 + \lambda h)e_n + [-y(x_{n+1}) + (1 + \lambda h)y(x_n) \qquad (7) \\ &= \text{Propagated Error} + \text{Local Truncation Error.} \end{aligned}$$