More about the RK Methods

- A R-stage RK method involves R function evaluations per step.
  - ◇ Each of the functions k_r, $r = 1, \ldots, R$, may be interpreted as approximation to the derivative $y'(x)$.
  - ◇ The function ϕ is a weighted mean of these approximations.
- If the B matrix in the Butcher array is *strictly lower triangular*, the resulting method is said to be *explicit*. In this case, the Runge-Kutta method is self-sufficient.
  - ◇ If an error estimator is available, then a variable-step Runge-Kutta method can easily be programmed becaused it is a one-step method.
  - ◇ RKF45 is a suggested method is many available ODE packages.
- Recall the Taylor series

$$y(x_{n+1}) = y(x_n) + hy^{(1)}(x_n) + \frac{h^2}{2!}y^{(2)}(x_n) + \ldots.$$

  - ◇ By defining

$$\phi_T(x, y, h) := f(x, y) + \frac{h}{2!}\frac{df}{dx}(x, y) + \ldots,$$

    the Taylor series can be thought of as a one-step method.
  - ◇ One of the main task in developing a Runge-Kutta method is to choose values for the constants c_r, a_r, b_{rs} so that the expansion of the function ϕ defined earlier in the Runge-Kutta method agrees with the expansion for $\phi_T(x, y)$ in as many terms as possible.
  - ◇ If the highest term matched has p in the power of h, we say the Runge-Kutta method is of order p. That is, a Runge-Kutta method is a *smart* way of implementing the Taylor series expansion.

◇ It is quite often the case that given a specific order, there are a 2-parameter family of Runge-Kutta methods all of which have the same order of accuracy.

- More Examples

◇ Two fourth-order 4-stage explicit Runge-Kutta methods

0	0			
1/2	1/2	0		
1/2	0	1/2	0	
1	0	0	1	0
	1/6	2/6	2/6	1/6

0	0			
1/3	1/3	0		
2/3	-1/3	1	0	
1	1	-1	1	0
	1/8	3/8	3/8	1/8

◇ The unique 2-stage implicit Runge-Kutta method of order 4:

$1/2+\sqrt{3}/6$	$1/4$	$1/4+\sqrt{3}/6$
$1/2-\sqrt{3}/6$	$1/4-\sqrt{3}/6$	$1/4$
	$1/2$	$1/2$

◇ A 3-stage semi-explicit Runge-kutta method or order 4:

0	0	0	0
1/2	1/4	1/4	0
1	0	1	0
	1/6	4/6	1/6