Stability Issue of RK Methods

- Apply a Runge-Kutta method to the test problem $y' = \lambda y$.
  - ⋄ Define the global error $\tilde{e}_n := y(x_n) - y_n$.
  - ⋄ Denote $\overline{h} := \lambda h, k = [k_1, \ldots, k_R]^T$ and $E := [1, \ldots, 1]^T$.
  - ⋄ Observe that
    $$k = \lambda E y_n + \overline{h} B k,$$
    or equivalently,
    $$k = \lambda (I - \overline{h} B)^{-1} E y_n.$$
  - ⋄ It follows that
    $$\begin{aligned} e_{n+1} &= e_n + \overline{h} c^T (I - \overline{h} B)^{-1} E e_n \\ &+ \text{(local error due to truncation and round-off)}. \end{aligned}$$
  - ⋄ The growth factor is
    $$r := 1 + \overline{h} c^T (I - \overline{h} B)^{-1} E.$$
  - ⋄ The Runge-Kutta method is said to be *absolutely stable* on the interval (α, β) if $|r| < 1$ whenever $\overline{h} \in (\alpha, \beta)$.
  - ⋄ For $R = 1, 2, 3, 4$, all R-stage explicit Runge-Kutta methods of order R have the same interval of absolute stability. In each of these cases,
    $$r = 1 + \overline{h} + \frac{1}{2}\overline{h}^2 + \ldots + \frac{1}{R!}\overline{h}^R. \tag{1}$$

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Examples on the Stability of RK Methods

- The explicit Euler method is a RK method with the Butcher array $\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$. The growth factor is given by

$$\rho = 1 + \overline{h}.$$

  - ◇ If $\lambda > 0$, then $\rho > 0$ whenever $h > 0$. That is, for exponentially growing solution, the explicit Euler method can *never* be stable in forward time.
  - ◇ If $\lambda < 0$, then the method is stable only if $0 < h < \frac{2}{|\lambda|}$. If $|\lambda|$ is very large, i.e., if the solution decays very rapidly, the step size h has to be small enough to guarantee the stability. This is the so called *still* problem.

- The implicit Euler method is a RK method with the Butcher array $\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$. The growth factor is given by

$$\rho = 1 + \overline{h}(1 - \overline{h})^{-1} = \frac{1}{1 - \lambda h}.$$

  - ◇ Again, the method cannot be stable in forward time when $\lambda > 0$.
  - ◇ It is important to note that when $\lambda < 0$, *any* positive step size h will make $\rho < 1$. This is a very welcomed feature called *A-stable*. The implicit Euler method is an A-stable method. An A-stable method is an ideal method for solving stiff ordinary differential equations.

- Any 2-stage explicit RK method will have the Butcher array $\begin{array}{c|cc} 0 & 0 & 0 \\ a_1 & a_2 & 0 \\ \hline & c_1 & c_2 \end{array}$

  where $c_1 + c_2 = 1$ and $c_2 a_2 = \frac{1}{2}$. The growth factor is given by

$$\rho = 1 + \overline{h}[c_1, c_2] \begin{bmatrix} 1 & 0 \\ -\overline{h}a_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 + \overline{h} + \frac{\overline{h}^2}{2}.$$