Multi-step Methods

- Let $x_0 < x_1 \ldots$ be a sequence of points at which the solution $y(x)$ to a differential equation $y' = f(x, y)$ is approximated by the values $y_1, y_1, \ldots$.
  - ◇ Any numerical method that computes y_{i+1} at x_{i+1} by using *information* at $x_i, x_{i-1}, \ldots, x_{i-k+1}$ is called a *k-step method*.
  - ◇ By a *linear* $(p+1)$-step method of step size h, we mean a numerical scheme of the form

$$y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i} + h \sum_{i=-1}^{p} b_i f_{n-i} \tag{1}$$

    where $x_k = x_0 + kh$, $f_{n-i} := f(x_{n-i}, y_{n-i})$, and $a_p^2 + b_p^2 \neq 0$.
    - ▷ Note that the information used involves the approximate solution y_i and its first order derivative f_i.
    - ▷ If $b_{-1} = 0$, then the method is said to be explicit; otherwise, it is implicit.
    - ▷ In order to obtain y_{n+1} from an implicit method, usually it is necessary to solve a nonlinear equation. Such a difficulty quite often is compensated by other more desirable properties which are missing from explicit methods. One such a desirable property is the stability.
- The *local truncation error* of the linear multi-step method at x_{n+1} is defined to be $L[y_{n-p}, h]$ where, with $a_{-1} = -1$,

$$L[y(x), h] := \sum_{i=-1}^{p} a_i y(x + (p-i)h) + h \sum_{i=-1}^{p} b_i y'(x + (p-i)h). \tag{2}$$

- To derive meaningful multi-step methods, one way is to expand the right hand side of (**??**) about x by the Taylor series and collect the like powers of h.

◇ We should have

$$L[y(x), h] = c_0 y(x) + c_1 h y^{(1)}(x) + \ldots + c_q h^q y^{(q)}(x) + \ldots \quad (3)$$

where

$$\begin{aligned} c_0 &= \sum_{i=-1}^{p} a_i, \\ c_1 &= \sum_{i=-1}^{p-1} (p-i) a_i + \sum_{i=-1}^{p} b_i, \\ c_q &= \frac{1}{q!} \sum_{i=-1}^{p-1} (p-i)^q a_i + \frac{1}{(q-1)!} \sum_{i=-1}^{p-1} (p-i)^{q-1} b_i, q \geq 2. \end{aligned}$$

◇ A linear multi-step method is said to be *of order* r if $c_0 = c_1 = \ldots = c_r = 0$, but $c_{r+1} \neq 0$, i.e., if $L[y(x), h] = c_{r+1} y^{(r+1)}(\eta) h^{r+1}$.

- Another way to derive meaningful multi-step method is to integrate both sides of the differential equation. Suppose we integrate from x_n to x_{n+1}, we obtain

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x)) dx. \quad (4)$$

◇ Suppose $(x_n, y_n), (x_{n-1}, y_{n-1}), \ldots, (x_{n-p}, y_{n-p})$ are already known. We may approximate $f(x, y(x))$ by a p-th degree interpolation polynomial $p_p(x)$. In this way, we obtain an explicit scheme

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} p_p(x) dx = y_n + \sum_{i=0}^{p} \beta_{pi} f_{n-i} \quad (5)$$

where b_i are the quadrature coefficients. Such a method is called an *Adams-Bashforth method*.

◇ Similarly, if we also include (x_{n+1}, y_{n+1}) in the data of interpolation, then we end up with an implicit scheme

$$y_{n+1} = y_n + \sum_{i=-1}^{p} \beta_{pi} f_{n-i} \quad (6)$$

which is called an *Adams-Moulton method*.

- Some Adams-Bashforth methods:

	0	1	2	3	4
β_{0i}	1				
$2\beta_{1i}$	3	-1			
$12\beta_{2i}$	23	-16	5		
$24\beta_{3i}$	55	-59	37	-9	
$720\beta_{4i}$	1901	-2774	2616	-1274	251

- Some Adams-Moulton methods:

	-1	0	1	2	3
β_{0i}	1				
$2\beta_{1i}$	1	1			
$12\beta_{2i}$	5	8	-1		
$24\beta_{3i}$	9	19	-5	1	
$720\beta_{4i}$	251	646	-264	106	-19