Example of Stability Issues in MS methods

- Consider the midpoint rule
$$y_{n+1} = y_{n-1} + 2hf_n$$
applied to the initial value problem
$$y' = -y, y(0) = 1.$$
  - ◇ The exact solution is $y(x) = e^{-x}$.
  - ◇ The numerical scheme produces a finite difference equation
  $$y_{n+1} = y_{n-1} - 2hy_n.$$
    - ▷ Try a solution of the form
    $$y_n = r^n.$$
    - ▷ The value of r must be zeros of the polynomial $r^2+2hr-1 = 0$.
    - ▷ The general solution is given by
    $$y_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$
    with $r_i = -h \pm \sqrt{h^2+1}$.
- Initial condition implies $\alpha_1 + \alpha_2 = 1$. We need one more value y_1 to start the midpoint rule.
- Expressing α_1 and α_2 in terms of y_1, we have
$$\alpha_1 = \frac{r_2 - y_1}{r_2 - r_1} = \frac{\sqrt{h^2+1} + h + y_1}{\sqrt[2]{h^2+1}} \tag{1}$$
$$\alpha_2 = \frac{r_1 - y_1}{r_1 - r_2} = \frac{-h + \sqrt{h^2+1} - y_1}{\sqrt[2]{h^2+1}}. \tag{2}$$
  - ◇ Note that $|r_1| < 1$ and $|r_2| > 1$.
  - ◇ Unless y_1 is such that $\alpha_2 = 0$, y_n grows unboundedly and, hence, deviated from the exact solution regardless of the step size.
- Such a phenomenon is called *numerical instability.*