Stability Issues in MS Methods

- Consider a general finite difference equation

$$\sum_{i=-1}^{p} a_i y_{n-i} = \phi_n \tag{1}$$

where a_i are constants independent of n, $a_{-1} \neq 0$, $a_p \neq 0$, and $n \geq p+1$.

  - ◇ Given starting values $y_0, \ldots, y_p$, a solution of the difference equation is a sequence of value $\{y_n\}_{p+1}^{\infty}$ that satisfies the equation (**??**) identically for all $n = p+1, \ldots$.
  - ◇ The general solution of (**??**) can be written as $y_n = \hat{y}_n + \psi_n$ where $\hat{y}_n$ is the general solution of the homogeneous equation

$$\sum_{i=-1}^{p} a_i y_{n-1} = 0 \tag{2}$$

    and ψ_n is a particular solution of (**??**).
  - ◇ To determine $\hat{y}_n$, we try $\hat{y}_n = r^n$ for some appropriate r. Then we find r must be a root of the polynomial

$$P(r) := \sum_{i=-1}^{p} a_i r^{p-i} = a_{-1} r^{p+1} + a_0 r^p + \ldots + a_p. \tag{3}$$

- Let the roots of $P(r)$ be denoted by $r_0, r_1, \ldots, r_p$.

  - ◇ Suppose all the roots of $P(r)$ are distinct. Then $r_j^n, j = 0, \ldots, p$ are linearly independent.
    - ▷ Linear independence means that

$$\sum_{j=0}^{p} \alpha_j r_j^n = 0 \text{ for all } n \Longrightarrow \alpha_j = 0 \text{ for all } j.$$

In this case, the general solution $\hat{y}_n$ of (??) is given by

$$\hat{y}_n = \sum_{j=0}^{p} \alpha_j r_j^n \tag{4}$$

where α_j are arbitrary constants to be determined by the starting values.

- ◇ Suppose some or all the roots of $P(r)$ are repeated. For example, suppose $r_0 = r_1 = r_2$ and suppose the remaining roots are distance. Then it can be proved that $r_0^n, nr_1^n, n(n-1)r_2^n, r_3^n, \ldots, r_p^n$ are linearly independent. In this case, the general solution $\hat{y}_n$ is given by

$$\hat{y}_n = \alpha_0 r_0^n + \alpha_1 n r_1^n + \alpha_2 n^2 r_2^n + \sum_{j=3}^{p} \alpha_j r_j^n. \tag{5}$$

- Note that $\hat{y}_n$ is bounded as $n \to \infty$ if and only if all roots of $P(r) = 0$ have modulus $|r_j| \leq 1$ and those with modulus 1 are simple.

- Corresponding to a given multi-step method, we denote

$$P_1(r) := \sum_{i=-1}^{p} a_i r^{p-i}, \tag{6}$$

$$P_2(r) := \sum_{i=-1}^{p} b_i r^{p-i}., \tag{7}$$

respectively, as the first and the second characteristic polynomials.

- Apply the multi-step method to the problem $y' = 0$.
  - ◇ We say the method is *zero-stable* if and only if all roots of $P_1(r)$ have modulus ≤ 1 and those with modulus 1 are simple.
  - ◇ In a $(p+1)$-step method, totally there are $2p+3$ undetermined coefficients if $a_{-1} = -1$.
  - ◇ We may choose these numbers so that $c_0 = \ldots = c_{2p+2} = 0$. Therefore, the maximal order for an implicit method is $2p+2$.
  - ◇ However, the maximal order method is often useless because of the well known Dalquist barrier theorem:

  - ▷ Any $(p+1)$-step method of order $\geq p+3$ is not zero stable. More precisely, if p is odd, then the highest order of a zero stable method is $p+3$; if p is even, then the highest order of a zero stable method is of order $p+2$.

- Apply the multi-step method to the problem $y' = \lambda y$.
  - ◇ Recall that the local truncation error T_{n+1} at x_{n+1}:

$$T_{n+1} := \sum_{i=-1}^{p} a_i y(x_{n-i}) + h \sum_{i=-1}^{p} b_i \lambda y(x_{n-i}). \tag{8}$$

  - ◇ Due to the floating-point arithmetic, we often will introduce round-off errors in applying the scheme. Suppose

$$R_{n+1} = \sum_{i=-1}^{p} a_i \tilde{y}_{n-i} + h \sum_{i=-1}^{p} b_i f(x_{n-i}, \tilde{y}_{n-i}). \tag{9}$$

  - ◇ The global errors $\tilde{e}_n := y(x_n) - \tilde{y}_{n-i}$ satisfy the difference equation

$$\sum_{i=-1}^{p} a_i \tilde{e}_{n-i} + \lambda h \sum_{i=-1}^{p} b_i \tilde{e}_{n-i} = T_{n+1} - R_{n+1}. \tag{10}$$

  - ◇ It is reasonable to assume that $T_{n+1} - R_{n+1} \approx \phi \equiv$ constant.
  - ◇ A general solution of (**??**) would be

$$\tilde{e}_n = \sum_{j=0}^{p} \alpha_j r_j^n + \frac{\phi}{\lambda h \sum_{i=-1}^{p} b_i}, \tag{11}$$

    if we assume all roots $r_0, \ldots, r_p$ of the polynomial $\sum_{i=-1}^{p}(a_i + \lambda h b_i) r^{p-i}$ are distinct.
  - ◇ The polynomial

$$\prod(r, \overline{h}) := P_1(r) + \overline{h} P_2(r) = \sum_{i=-1}^{p} (a_i + \overline{h} b_i) r^{p-i} \tag{12}$$

    is called the *stability polynomial* of the multi-step method