Bisection Method

- Find a root for a equation $f(x) = 0$ is an important takes occurred in almost every branch of scientific and engineering applications.
  - ◇ The function may be linear or nonlinear.
  - ◇ The function may be smooth or non-smooth.
  - ◇ The equation may not have a solution — existence question.
  - ◇ The solutions, when they exist, may not be unique.
  - ◇ The solution may have to be obtained by numerical method.
- Bisection Method:
  - ◇ If f is continuous on $[a, b]$ and c is a value lies between $f(a)$ and $f(b)$, then there exists a point $x \in [a, b]$ such that $f(x) = c$.
  - ◇ If $\text{sign}(f(a)) \neq \text{sign}(f(b))$, then $f(x) = 0$ has at least one solution that lies between a and b.
  - ◇ Take $c = \frac{a+b}{2}$ and check the value of $f(c)$.
    - ▷ If $f(c) = 0$, the c is a root.
    - ▷ If $f(c) \neq 0$ and $\text{sign}(f(c)) \neq \text{sign}(f(b))$, then a root lies between c and b.
    - ▷ If $f(c) \neq 0$ and $\text{sign}(f(c)) \neq \text{sign}(f(a))$, then a root lies between c and a.
    - ▷ Repeat the process until the root lies in an interval of length less than a prescribed ϵ.
  - ◇ The hardest part about using the bisection method is to find the first interval $[a, b]$. After that, the method is guaranteed to converge steadily.
  - ◇ If $L_0 = |b - a|$, then for the algorithm to stop when $L_k \leq \epsilon$, it will require $\log_2 \frac{L_0}{\epsilon}$ iterations to converge.

Newton's Method

- Newton's Method:
  - ◇ Starting from an initial guess x_0 of the true root x^*, Newton's method generates successive approximations $x_1, x_2, \ldots$ according to the scheme

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, \ldots. \tag{1}$$

  - ◇ Newton's method can be derived in two ways: geometrically and analytically.
    - ▷ Geometrically, x_{k+1} is the x-intercept of the line that is tangent to the graph of $f(x)$ at $(x_k, f(x_k))$.
    - ▷ Analytically, $f(x_{k+1})$ is the linear approximation to $f(x)$ in the neighborhood of x^*.
- Newton's method can be generalized to systems of linear equations. In that case, the scheme goes as follows:

$$f'(x_k)\Delta x_k = -f(x_k), \tag{2}$$
$$x_{k+1} = x_k + \Delta x_k. \tag{3}$$

- Two examples:
  - ◇ Suppose $a > 0$ and $f(x) = \frac{1}{x} - a$. A Newton iteration for finding the reciprocal of a takes the scheme

$$x_{k+1} = 2x_k - ax_k^2.$$

  - ◇ Suppose $a > 0$ and $f(x) = x^2 - a$. A Newton iteration for finding the square root of a takes the scheme

$$x_{k+1} = \frac{1}{2}(x_k + \frac{a}{x_k}).$$