Convergence Analysis

- Assume x_0 is near enough to a zero x_* of the function $f(x)$ and $f^{'}(x^*) \neq 0$.
- Define $e_k = x_k - x^*$. The convergence analysis involves three steps:
  - ◇ Obtain an expression for e_{k+1} in terms of e_k.
  - ◇ Use the expression to show that $e_k \longrightarrow 0$.
  - ◇ Assess how fast e_k converges to zero.
- Define the iteration function $\phi(x)$ for Newton's method by

$$\phi(x) := x - \frac{f(x)}{f'(x)}.$$

  Observe the following two iterations.

  - ◇ A Newton step is equivalent to $x_{k+1} = \phi(x_k)$.
  - ◇ At the zero of f, $x^* = \phi(x^*)$.
- Error Formula:
  - ◇ $e_{k+1} = \phi(x_k) - \phi(x^*)$.
  - ◇ $e_{k+1} = \phi^{'}(\xi_k)e_k$ for some ξ_k between x_k and x^*.
  - ◇ $\phi^{'}(x) = \frac{f(x)f^{''}(x)}{(f^{'}(x))^2}$.
    - ▷ Since $f(x^*) = 0$, it follows that $\phi^{'}(x^*) = 0$.
    - ▷ By continuity, there is an interval $I_\delta = [x^* - \delta, x^* + \delta]$ about x^* such that

$$|\phi^{'}(x)| \leq C < 1 \text{ whenever } x \in I_\delta.$$

- ⋄ Suppose $x_0 \in I_\delta$. Then

$$|e_1| \leq |\phi^{'}(\xi_0)||e_0| \leq C|e_0| \leq C\delta < \delta,$$

and $x_1 \in I_\delta$.

- ⋄ Repeat by the same reasoning, we find that

$$|e_k| \leq C|e_{k-1}| \leq C^k|e_0|.$$

This proves the convergence.

- Quadratic Convergence:
  - ⋄ Since $\phi^{'}(x^*) = 0$, we can write

$$\phi(x_k) - \phi(x^*) = \frac{1}{2}\phi^{''}(\eta_k)(x_k - x^*)^2.$$

where η_k lies between x_k and x^*.

  - ⋄ It follows

$$e_{k+1} = \frac{1}{2}\phi^{''}(\eta_k)e_k^2. \tag{1}$$

  - ⋄ Since η_k approaches x^* along with x_k, it follows that

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \frac{1}{2}\phi^{''}(x^*) = \frac{f^{''}(x^*)}{2f^{'}(x^*)}.$$

That is,

$$e_{k+1} \approx \frac{f^{''}(x^*)}{2f^{'}(x^*)}(\eta_k)e_k^2. \tag{2}$$

The Newton's method is preferred because of this *quadratic convergence* property.