Fundamentals

- Basic Operations:
  - ◇ Matrix-vector product $b = Ax$, where $A \in R^{m \times n}$ and $x \in R^n$, is defined by
  $$b_i = \sum_{j=1}^{n} a_{ij} x_j,\ i = 1, \ldots, m.$$
  - ◇ Matrix-matrix product $B = AC$, where $A \in R^{\ell \times m}$ and $C \in R^{m \times n}$, is defined by
  $$b_{ij} = \sum_{k=1}^{m} a_{ik} c_{kj},\ i = 1, \ldots, \ell,\ j = 1, \ldots, n.$$
  - ◇ Outer product: Let the columns of A be denoted as $A = [A_1, \ldots, A_n]$. Then $b = Ax$ can be written as
  $$b = \sum_{j=1}^{n} A_j x_j,$$
  i.e., b is a linear combination of columns of A.
- Basic Definitions: Let $A \in R^{m \times n}$. Then
  - ◇ $R(A)$ = range space of $A = \{Ax \in R^m | x \in R^n\}$.
  - ◇ $N(A)$ = null space of $A = \{x \in R^n | Ax = 0\}$.
    - ▷ Note that both $R(A)$ and $N(A)$ are linear subspaces. Thus they have bases and dimensionalities.
  - ◇ $rank(A)$ = the dimension of $R(A)$.
    - ▷ *Matrix A has full rank if and only if $m \geq n$ and A is a one-to-one map.*

- Inverse and Nonsingularity:
  - ⋄ The operation Ax can be considered as a linear map from R^n to R^m.
  - ⋄ Suppose A is a square matrix. The following results are equivalent:
    - ▷ A has an inverse A^{-1}.
    - ▷ $rank(A) = n$.
    - ▷ $R(A) = R^n$.
    - ▷ $N(A) = \{0\}$.
    - ▷ $\det(A) \neq 0$.
    - ▷ $Ax = b$ has a unique solution for every b.
    - ▷ All n columns of A are linearly independent.
- Orthogonal Vectors:
  - ⋄ We say two vectors $x, y \in R^n$ are *orthogonal* to each other if and only if $x^T y = 0$.
  - ⋄ *The vectors in an orthogonal set are linearly independent.*
  - ⋄ A square matrix $A \in R^{n\times n}$ is called an orthogonal matrix if all its columns are *orthonormal*, i.e., if $A_i^T A_j = \delta_{ij}$.
    - ▷ A is orthogonal if and only if $A^T A = I$, i.e., $A^{-1} = A^T$.
- Norms:
  - ⋄ The essential notions of size and distance in a vector space are captured by *norms*.
  - ⋄ Suppose $x \in R^n$. Then following are some commonly used vector norms for x.

$$
\begin{aligned}
\|x\|_1 &:= \sum_{i=1}^{n} |x_i|, \\
\|x\|_2 &:= \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \\
\|x\|_\infty &:= \max_{1\leq i\leq n} |x_i|, \\
\|x\|_p &:= \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \ (1 \leq p < \infty).
\end{aligned}
$$