Matrix Norm

- One of the main concerns in matrix algorithms is the sensitivity of a system to its coefficients. To do so, we need to measure the size of errors. The measurement is done by the notion of *norm*.
- We have already seen some examples of vector norms. Mathematically, a *vector norm* is a function $\|\cdots\| : R^n \longrightarrow R$ that satisfies:
  1. $x \neq 0 \Longrightarrow \|x\| > 0$,
  2. $\|\alpha x\| = |\alpha| \|x\|$,
  3. $\|x + y\| \leq \|x\| + \|y\|$.
- *Matrix norms* are defined in analogy with vector norms because a matrix in $R^{n \times n}$ can be thought of as a vector in R^{n^2}. There are additional properties (and often confusing), however, in matrix norms.
  - ◇ We say a matrix norm $\|\cdot\|$ is *consistent* if it satisfies
    $$\|AB\| \leq \|A\| \|B\| \tag{0.1}$$
    whenever AB is defined.
    - ▷ Not all matrix norms are consistent. For example, if we define $\|A\| = \max_{i,j} |a_{ij}|$ like the sup-norm for vectors, we find
      $$\left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\| = 2,$$
      while
      $$\left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\| = 1.$$
  - ◇ The *induced* norm
    $$\|A\| := \max_{x \neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}} \tag{0.2}$$
    where $\|\cdot\|_{(m)}$ and $\|\cdot\|_{(n)}$ are fixed vector norms is a consistent matrix norm.