Solving Triangular Systems

- A matrix $L = [\ell_{ij}]$ is *lower triangular* if $\ell_{ij} = 0$ whenever $i < j$.
  - ⋄ If the diagonal elements ℓ_{ii} of L are nonzero, the L is nonsingular.
  - ⋄ The linear equation $Lx = b$ where L is a nonsingular lower triangular matrix can be solved by *forward substitutions*.
- Forward Substitution Algorithm can be sufficiently illustrated by the case $n = 5$.

$$\begin{aligned} b_1 &= \ell_{11}x_1 \\ b_2 &= \ell_{21}x_1 + \ell 22x_2 \\ b_3 &= \ell_{31}x_1 + \ell 32x_2 + \ell 33x_3 \\ b_4 &= \ell_{41}x_1 + \ell 42x_2 + \ell 43x_3 + \ell 44x_4 \\ b_5 &= \ell_{51}x_1 + \ell 52x_2 + \ell 53x_3 + \ell 54x_4\ell_{55}x_5 \end{aligned}$$

  - ⋄ The first equation involves x_1 only. So

$$x_1 = \frac{b_1}{\ell_{11}}.$$

  - ⋄ Knowing x_1, we can substitute it into the second equation and solve to get

$$x_2 = \frac{b_2 - \ell_{21}x_2}{\ell_{22}}.$$

  - ⋄ Substituting x_1 and x_2 into the third equation and solving, we get

$$x_3 = \frac{b_3 - \ell_{31}x_1 - \ell_{21}x_2}{\ell_{33}}.$$

  - ⋄ Continuing in this manner, we get the general idea of solving for x_i,

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} \ell_{ij}x_j}{\ell_{ii}}.$$

- ◇ The total number of multiplications/divisions involved in the forward substitution algorithm is $\sum_{i=1}^{n}\sum_{j=1}^{i} = \frac{n(n+1)}{2}$. Similarly, there are a like number of additions/subtractions.

- An upper triangular system $Ux = b$ can be solved by *backward substitutions.*

- Suppose a square matrix A can be factorized into the form

$$A = LU \tag{0.1}$$

  where L is a lower triangular matrix and U is an upper triangular matrix. Then we can write the linear equation $Ax = b$ in the form $LUx = b$. The system can be solved in two states:

  1. First, we solve the lower triangular system

  $$Ly = b.$$

  2. Secondly, we solve the upper triangular system

  $$Ux = y,$$

     where y is obtained from the first state.

  The overall cost, given the LU factorization, is $O(n^2)$. This is the main idea of *Gaussian Elimination Method.* It remains to see how the LU decomposition can be done.