Gaussian Elimination

- Consider the Consider the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{12}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n, \end{cases}$$

  Gauss elimination method consists in

  - ◇ Using the first equation (assuming $a_{11} \neq 0$) to eliminate all x_1 terms from the second equation on;
  - ◇ Then using the new second equation to eliminate all x_2 terms from the third equation on;
  - ◇ Until in the end, we obtain a new system

$$\begin{cases} \overline{a}_{11}x_1 + \overline{a}_{12}x_2 + \ldots + \overline{a}_{1n}x_n = \overline{b}_1 \\ \qquad\quad \overline{a}_{22}x_2 + \ldots + \overline{a}_{2n}x_n = \overline{b}_2 \\ \qquad\qquad\qquad\qquad\qquad\quad \vdots \\ \qquad\qquad\qquad\qquad \overline{a}_{nn}x_n = \overline{b}_n \end{cases}$$

- The elimination process can be summarized as follows:

```
for k = 1:n
    if a(k,k} == 0,
       quit
   else
      w(j} = a(k,j}
   for i = k+1:n
      eta = a(i,k)/a(k,k)
      a(i,k) = eta
      for j = k+1:n
         a(i,j) = a(i,j) - eta * w(j)
      end
   end
end
```

- ◇ The above algorithm computes the factorization $A = LU$.
- ◇ The strictly lower triangular portion of A is overwritten by elements of L whose diagonal elements are constantly 1.
- ◇ The upper triangular portion of A is overwritten by elements of U.
- ◇ The algorithm will terminate prematurely if it encounters a zero *pivot* element.

- In general, after $k-1$ steps, we should have constructed the matrix

$$A^{(k)} := \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & & & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & & & & a_{2n}^{2} \\ & \ddots & \ddots & & & \vdots \\ 0 & & 0 & a_{kk}^{(k)} & \dots & a_{kn}^{(k)} \\ & & & & \vdots & \\ 0 & & 0 & a_{nk}^{(k)} & \dots & a_{nn}^{(k)} \end{bmatrix}$$

- ◇ Assuming $a_{kk}^{(k)} \neq 0$, we define the multipliers $m_{ik} := a_{ik}^{(k)}/a_{kk}^{(k)}$ for $i = k+1, \dots, n$.
- ◇ Then we generate $a_{ij}^{(k+1)} := a_{ij}^{(k)} - m_{ik}a_{kj}^{(k)}$ for all $i, j = k+1, \dots, n$.
- ◇ In do so, the earlier k rows are left unchanged, and zeros are introduced into column k below the diagonal element.

- The elimination process can be described in terms of elementary matrix operations:

- ◇ Define

$$\begin{array}{r} E_{ji}(p) := \\ j\text{-th row} \longrightarrow \end{array} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & p & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

$$\uparrow$$
i-th column

- ▷ $E_{ji}(p)A$ amounts to multiplying the i-th row of A by the scalar p and then adding the result to the j-th row of A.
- ▷ $\det E_{ji}(p) = 1$.
- ▷ $(E_{ji}(p))^{-1} = E_{ji}(-p)$.

◇ The transformation from $A^{(1)}$ to $A^{(2)}$ can be summarized as

$$E_{n1}(-m_{n1}) \dots E_{21}(-m_{21})A^{(1)} = A^{(2)}.$$

◇ Likewise, $A^{(2)}$ can be transformed to $A^{(3)}$ through

$$E_{n2}(-m_{n2}) \dots E_{32}(-m_{32})A^{(2)} = A^{(3)}.$$

◇ In the end, we have

$$\underbrace{E_{n,n-1}E_{n-1,n-2} \dots E_{n2} \dots E_{32}E_{n1} \dots E_{21}}_{L^{-1}} A^{(1)} = A^{(n)},$$

where $A^{(n)}$ is an upper triangular matrix, and L^{-1}, being the product of a sequence of lower triangular matrices, is a nonsingular lower triangular matrix.

◇ It remains to check out what L^{-1} is.