LU Decomposition

- We have seen that the Gaussian elimination process can be described in terms of elementary matrix operations:

$$\begin{aligned} E_{n1}(-m_{n1})\dots E_{21}(-m_{21})A^{(1)} &= A^{(2)}, \\ E_{n2}(-m_{n2})\dots E_{32}(-m_{32})A^{(2)} &= A^{(3)}, \\ &\vdots \\ E_{n,n-1}(-m_{n,n-1})A^{(n-1)} &= A^{(n)}. \end{aligned}$$

- It is easy to see that

$$\begin{aligned} L &:= \{E_{n,n-1}E_{n-1,n-2}\dots E_{n2}\dots E_{32}E_{n1}\dots E_{21}\}^{-1} \\ &= \begin{bmatrix} 1 & 0 & \dots & & & \\ m_{21} & 1 & & & & \\ & m_{22} & & & & \\ \vdots & \vdots & & & & \\ & & & & & \\ m_{n-1,1} & & & & 1 & 0 \\ m_{n1} & m_{n2} & & & m_{n,n-1} & 1 \end{bmatrix}. \end{aligned} \tag{0.1}$$

- It follows that if the Gaussian elimination process does not terminate prematurely, then A can be decomposed into the product of a lower triangular matrix, i.e., L, and and an upper triangular matrix, i.e., $A^{(n)}$. This is called the *LU decomposition* of A.

- The LU decomposition as described above can be carried out (without pivoting) if and only if $a_{kk}^{(k)} \neq 0$ for $k = 1, \dots, n-1$.

  ◇ When $a_{kk}^{(k)} = 0$, it is possible to perform the elimination by interchanging equations. Such a process of interchanging rows or columns to bring a nonzero element to the pivoting position is called *mathematical pivoting.*