Error Analysis

- Because of the floating-point arithmetic involved in the Gaussian elimination process, the linear system $Ax = b$ *cannot* be solved exactly. It can be shown by *backward error analysis* that the approximate solution y satisfies a perturbed system

$$(A + E)y = b + f$$

for some perturbations E and f. Sources of errors are

  - ◇ The round-off errors in representing A and b, say,

$$\begin{aligned} A_r &= A + \delta A, \\ b_r &= b + \delta b. \end{aligned}$$

  - ◇ The errors occurred in the decomposition of A_r, say

$$L_r U_r = P A_r + \delta A_r.$$

  - ◇ The floating-point arithmetic errors in solving the triangular systems.

- Assume that both A^{-1} and $(A + E)^{-1}$.

  - ◇ Clearly, $x = (A + E)^{-1}(B + Ex)$.
  - ◇ It follows $x - y = (A + E)^{-1}(Ex - F)$.
  - ◇ Taking any induced norm, we obtain

$$\|x - y\| \le \|(A + E)^{-1}\|(\|E\|\|x\| + \|F\|).$$

  - ◇ Note that $\|Ax\| = \|B\| \le \|A\|\|x\|$ and, hence, $\|B\|/\|A\| \le \|x\|$. It follows that

$$\begin{aligned} \frac{\|x - y\|}{\|x\|} &\le \|(A + E)^{-1}\|(\|E\| + \frac{\|F\|\,\|A\|}{\|B\|}) \\ &= \|(A + E)^{-1}\|\|A\|(\frac{\|E\|}{\|A\|} + \frac{\|F\|}{\|B\|}). \qquad (0.1) \end{aligned}$$

- (Banach Lemma) If D is an $n \times n$ matrix with $\|D\| < 1$, then $(I+D)^{-1}$ exists and satisfies

$$\|(I+D)^{-1}\| \leq \frac{1}{1-\|D\|}. \tag{0.2}$$

  - ◇ By the triangle inequality, we have

$$\|(I+D)x\| = \|x+Dx\| \geq \|x\| - \|Dx\| \geq (1-\|D\|)\|x\|$$

    for every x.
  - ◇ It follows that $\|(I+D)x\| > 0$ if $x \neq 0$. That is, $(I+D)x = 0$ has only the trivial solution $x = 0$. This shows $I+D$ is nonsingular.
  - ◇ Furthermore,

$$\begin{aligned} 1 &= \|I\| = \|(I+D)(I+D)^{-1}\| = \|(I+D)^{-1} + D(I+D)^{-1}\| \\ &\geq \|(I+D)^{-1}\| - \|D\|\|(I+D)^{-1}\| = \|(I+D)^{-1}\|(1-\|D\|). \end{aligned}$$

    This proves the assertion.

- We may estimate $\|(A+E)^{-1}\|$ as follows.

$$\begin{aligned} \|(A+E)^{-1}\| &= \|[A(I+A^{-1}E)]^{-1}\| \leq \|A^{-1}\|\|(I+A^{-1}E)^{-1}\| \\ &\leq \frac{\|A^{-1}\|}{1-\|A^{-1}E\|} \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|\|E\|} \end{aligned} \tag{0.3}$$

  provided $\|A^{-1}\|\|E\| < 1$.

- Together with the inequality (0.1), we can now conclude an error estimate for using the Gaussian elimination process.
  - ◇ Suppose $\|E\| < \frac{1}{\|A^{-1}\|}$.
  - ◇ Then

$$\frac{\|x-y\|}{\|x\|} \leq \frac{k(A)}{1-k(A)\frac{\|E\|}{\|A\|}}\left(\frac{\|E\|}{\|A\|} + \frac{\|F\|}{\|B\|}\right). \tag{0.4}$$

  - ◇ $k(A) := \|A\|\|A^{-1}\|$ is called the *condition number* of A with respect to the norm $\|\cdot\|$.