Some Examples

- The Hilbert matrix H_n of order n defined by

$$H_n := \begin{bmatrix} 1 & 1/2 & \dots & 1/n \\ 1/2 & 1/3 & \dots & 1/(n+1) \\ \vdots & & & \\ 1/n & 1/(n+1) & \dots & 1/(2n-1) \end{bmatrix}$$

can be inverted exactly and its eigenvalues can be computed exactly. However, the Hilbert matrix is very ill-conditioned.

n	3	4	5	6	7	8
$k(H_n)$	5.24×10^2	1.55×10^4	4.77×10^5	1.50×10^7	4.75×10^8	1.53×10^{10}

- Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$ and the equations $Ax = b$ with $b = [2, 2.0001]^T$ and $[2, 2.0002]^T$, respectively. The solutions are $[1, 1]^T$ and $[0, 2]^T$. This sensitivity of solution to small changes in data is related to the ill-conditioning of the matrix A.

- Let y denote an approximate solution to the system $Ax = b$. Define $r := Ay - b$ to be the *residue*. We normally expect that if r is small, then y would be close to the true solution x. But note that $y-x = A^{-1}r$ and, hence,

$$\frac{\|y-x\|}{\|x\|} = \frac{\|A^{-1}r\|}{\|x\|} \leq \frac{\|A^{-1}\|\|r\|}{\|x\|} = \frac{k(A)\|r\|}{\|A\|\|x\|} \leq \frac{k(A)\|r\|}{\|b\|}.$$

  ◇ Consider the system,

$$\begin{bmatrix} 0.780, & 0.563 \\ 0.913, & 0.659 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}.$$

  The exact solution is $[1, -1]^T$. Consider the two approximate solutions $y_1 = [0.341, -0.087]^T$ and $y_2 = [0.999, -1.001]^T$. Clearly y_2 should be a more reasonable solution. However, it can be checked that $r_1 = [-0.000010, 0.000000]^T$ and $r_2 = [-0.001343, -0.001572]^T$.