QR via Householder Transformation

- Let $u \in R^m$ be a column unit vector. The associated *Householder matrix* is defined to be

$$V := I - 2uu^T.$$

  - ◇ The matrix V is an orthogonal matrix.
  - ◇ The transformation

$$Vx = x - 2uu^T x.$$

    has a special meaning: It is the *reflection* of x with respect to the hyperplane normal to the vector u.

- The reflection property of the Householder transformation may be applied to construct the QR decomposition.

  - ◇ The idea: Given $x \in R^m$, find an Householder matrix V such that $Vx = \pm e\sigma$ where $\sigma = \|x\|_2$ (Why is such a quantity σ needed here?).
  - ◇ Two choices of u:

$$\begin{aligned} u_1 : &= \frac{\frac{x}{\sigma} + e_1}{\|\frac{x}{\sigma} + e_1\|_2}, \\ u_2 : &= \frac{\frac{x}{\sigma} - e_1}{\|\frac{x}{\sigma} - e_1\|_2}. \end{aligned} \tag{0.1}$$

  - ◇ Suppose the second choice is used, then $Vx = [\sigma, 0, \ldots, 0]^T$.

- Applied to $A \in R^{m\times n}$ with $m \geq n$, let x denote the first column of A.

  - ◇ There exists an orthogonal matrix $V_1 \in R^{m\times n}$ such that

$$A^{(2)} := V_1 A = \begin{bmatrix} \times & \times & \ldots & \times \\ 0 & \times & & \\ \vdots & & B^{(2)} & \\ 0 & \times & & \times \end{bmatrix}.$$

- By a similar argument, there exists an orthogonal matrix V_2' of size $(m-1)\times(m-1)$ such that

$$V_2' B^{(2)} = \begin{bmatrix} \times & \times & \dots & \times \\ 0 & \times & & \\ & & & \\ & & & \\ 0 & \times & & \times \end{bmatrix}.$$

◇ Now let $V_2 := \begin{bmatrix} 1 & 0 \\ 0 & V_2' \end{bmatrix} \in R^{m\times n}$. Then

$$V_2 A^{(2)} = \begin{bmatrix} \times & \times & \dots & & \times \\ 0 & \times & & & \\ & 0 & \times & & \times \\ & & & B^{(3)} & \\ 0 & 0 & \times & & \times \end{bmatrix}.$$

- Continuing this procedure $n-1$ times, we obtain

$$A^{(n)} := V_{n-1}A^{(n-1)} - V_{n-1}\dots V_1 A = \begin{bmatrix} \times & \times & \dots & \times \\ & 0 & & \times \\ & & \ddots & \\ & & & \times \\ 0 & & & 0 \\ & & & \\ 0 & \dots & & 0 \end{bmatrix} = R.$$

◇ Let

$$Q := V^{-1} := (V_{n-1}\dots V_1)^{-1}.$$

Then Q is also an orthogonal matrix.