Singular Value Decomposition

- The singular value decomposition (SVD) is a matrix factorization that serves both as the first computational step in many numerical algorithms and as the first conceptual step in many theoretical studies.
- Given $A \in R^{m\times n}$ $(m \geq n)$, the image of the unit sphere in R^n under A is a hpyerellipse (of dimension n) in R^m.
    - ◇ The unit vectors $u_i \in R^m$ in the directions of the principal semiaxes of the hyperellipse are called the *left singular vectors* of A.
    - ◇ The unit vectors $v_i \in R^n$ in the directions of the preimage of the principal semiaxes of the hyperellipse are called the *right singular vectors* of A.
    - ◇ The lengths σ_i of the principal semiaxes of the hyperellipse are called the *singular values* of A.
        - ▷ The above quantities are related to each other by the relationship:
        $$Av_i = \sigma_i u_i. \tag{0.1}$$
        - ▷ It is clear that all u_i, $i = 1, \ldots, n$ are mutually orthogonal.
        - ▷ It will be shown that all v_i, $i = 1, \ldots n$ are also mutually orthogonal.
- Recall that $A^TA \in R^{n\times n}$ is symmetric and positive semi-definite.
    - ◇ A^TA has a complete set of eigenvectors.
    - ◇ All eigenvalues of A^TA are nonnegative.
    - ◇ Denote the positive eigenvalues of A^TA by $\sigma_1^2 \geq \ldots \geq \sigma_r^2 > 0$.
        - ▷ It can be proved (but not done in this class) that $r = \text{rank}(A)$.
    - ◇ Denote the normalized (and orthogonal) eigenvector of A^TA associated with σ_i^2 by v_i

- Some important observations:
  - ◇ The two matrices A^TA and AA^T have the same positive eigenvalues.
  - ◇ Av_i is an eigenvector of AA^T associated with eigenvalue σ_i^2.
  - ◇ The vector $u_i := Av_i/\sigma_i$ is a normalized eigenvector of AA^T.
- The *singular value decomposition* of A:
  - ◇ Let $V := [v_1, \ldots, v_n] \in R^{n\times n}$ whose columns v_i are orthonormal eigenvectors of A^TA.
  - ◇ Define $U := [u_1, \ldots, u_m] \in R^{m\times m}$ where
    - ▷ For $j = 1, \ldots, r$, $u_j := Av_j/\sigma_j$, and
    - ▷ For $j = r+1, \ldots, m$, $\{u_{r+1}, \ldots, u_m\}$ are orthonormal eigenvectors corresponding to the zero eigenvalue of AA^T.
  - ◇ Define $\Sigma := \text{diag}\{\sigma_1, \ldots, \sigma_r\}$.
  - ◇ With U, Σ and V given above, it must be true that

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T. \tag{0.2}$$

    - ▷ Write $U = [U_1, U_2],\ V = [V_1, V_2]$.
    - ▷ Observe that

$$U^TAV = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} A[V_1, V_2] = \begin{bmatrix} U_1^TAV_1 & U_1AV_2 \\ U_2^TAV_1 & U_2^TAV_2 \end{bmatrix}.$$

    - ▷ Note that $AV_2 = 0$, $U_2^TAV_1 = U_2^TU_1\Sigma = 0$ and $U_1^TAV_1 = \Sigma$ by the choice of U.