SVD and Least Squares

- Consider the least squares problem

$$\min_{x \in R^n} \|Ax - b\|_2$$

where $A \in R^{m \times n}$ and $m \geq n$.

- Let the singular value decomposition of A be given by

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

- The vector $\tilde{x}$ given by

$$\tilde{x} := V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & \end{bmatrix} U^T b \tag{0.1}$$

is a solution to the least squares problem.

◇ Observe the facts

$$\begin{aligned} \|Ax - b\|_2^2 &= \|U^T(Ax - b)\|_2^2 = \|U^T AVV^T x - U^T b\|_2^2 \\ &= \| \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} z - c\|_2^2. \end{aligned}$$

▷ $z := V^T x = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

▷ $c := U^T b = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

◇ It follows $\|Ax - b\|_2^2 = \| \begin{bmatrix} \Sigma z_1 - c_1 \\ c_2 \end{bmatrix} \|_2^2 = \|\Sigma z_1 - c_1\|_2^2 + \|c_2\|_2^2$.

◇ $\|Ax - b\|$ is minimized if and only if $z_1 = \Sigma^{-1} c_1$.

▷ Note that z_2 can be arbitrary.

◇ Choose $\tilde{z} = \begin{bmatrix} \Sigma^{-1}c_1 \\ 0 \end{bmatrix}$ and define $\tilde{x} := V\tilde{x}$.

▷ Note

$$\begin{aligned} \tilde{x} &:= V\tilde{z} = V \begin{bmatrix} \Sigma^{-1}c_1 \\ 0 \end{bmatrix} \\ &= V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T b. \end{aligned}$$

- If $\overline{x}$ is another least square solution, then $\|\tilde{x}\|_2 \leq \|\overline{x}\|_2$.

◇ For any other least squares solution $\overline{x}$, the corresponding $\overline{z}$ must be of the form $\begin{bmatrix} \Sigma^{-1}c_1 \\ z_2 \end{bmatrix}$.

◇ Observe $\|\overline{x}\|_2^2 = \|U\overline{z}\|_2^2 = \|\overline{z}\|_2^2 = \|\Sigma^{-1}c_1\|_2^2 + \|z_2\|_2^2 \geq \|\tilde{z}\|_2^2 = \|\tilde{x}\|_2^2$.