# Eigenvalue Problems

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- Eigenvalue problems are particularly interesting in scientific computing because
  - ◇ Eigenvalue analysis is an important practice in many fields of engineering or physics.
  - ◇ Eigenvalue analysis play an important role in the performance of many numerical algorithms.
  - ◇ There are powerful algorithms for finding eigenvalues, yet these algorithms are far from obvious.
- What is a general eigenvalue problem?
  - ◇ Given $n \times n$ matrices $A$ and $B$, find numbers $\lambda$ such that the equation

    $$Ax = \lambda Bx \tag{0.1}$$

    is satisfied for some nontrivial vector $x \neq 0$.
  - ◇ If $B$ is invertible, then (0.1) can be reduced to

    $$Cx = \lambda x. \tag{0.2}$$

  - ◇ The nonzero vector $x$ is called an *eigenvector* and the corresponding scalar $\lambda$ is called an *eigenvalue* of the pair $(A, B)$ (or the matrix $C$.)
  - ◇ Generally, $\lambda$ and $x$ are complex-valued.
- Finding the solution of eigensystems is a fairly complicated procedure.
  - ◇ It is at least as difficult as finding the roots of polynomials.
  - ◇ Any numerical method for solving eigenvalue problems is expected to be iterative in nature. No direct method is available.
  - ◇ Algorithms for solving eigenvalue problems include the power method, subspace iteration, the QR algorithm, the Jacobi method, the Arnoldi method and the Lanczos algorithm.

- Spectral decomposition:

  ◇ If a matrix $A$ of size $n \times n$ has (a complete set of ) $n$ eigenvectors $x_1, \ldots, x_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then we may write
  $$AX = X\Lambda,$$
  where $X = [x_1, \ldots x_n]$ and $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$. If $X$ is invertible, the matrix $A$ then has a *spectral decomposition*
  $$A = X\Lambda X^{-1}. \tag{0.3}$$

  ◇ Note that not all matrices will have a spectral decomposition. If $A$ does have a spectral decomposition, we say $A$ is *diagonalizable.*

- Generically, *almost all* matrices are diagonalizable. But there are *defective* matrices. For example, the matrix $J$ where
  $$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$
  has eigenvalues $\lambda = 2$ with multiplicity 3, but has only one eigenvector corresponding to it. $J$ is not diagonalizable.

- If $X$ is nonsingular, then $A$ and $XAX^{-1}$ gave the same eigenvalues. A transformation by the relationship $XAX^{-1}$ is called a *similarity transformation.*

  ◇ To find eigenvalues of a matrix $A$, it suffices to transform $A$ by similarity transformations to an upper triangular matrix.

  ◇ Schur decomposition: Every square matrix $A$ can be transformed by unitary similarity transformation into an upper triangular matrix, i.e., there exist an unitary matrix $Q \in C^{n\times n}$ such that
  $$A = QTQ^*, \tag{0.4}$$
  where $T$ is upper triangular. This is the principal basis of numerical algorithms for eigenvalue computation.